THE HÖLDER PROPERTY FOR THE SPECTRUM OF TRANSLATION FLOWS IN GENUS TWO

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ABSTRACT. The paper is devoted to generic translation flows corresponding to Abelian differentials with one zero of order two on flat surfaces of genus two. These flows are weakly mixing by the Avila-Forni theorem. Our main result gives first quantitative estimates on their spectrum, establishing the Hölder property for the spectral measures of Lipschitz functions. The proof proceeds via uniform estimates of twisted Birkhoff integrals in the symbolic framework of random Markov compacta and arguments of Diophantine nature in the spirit of Salem, Erdős and Kahane.

To the memory of William Austin Veech (1938–2016)

1. INTRODUCTION

1.1. **Spectrum of translation flows.** Let M be a compact orientable surface. To a holomorphic one-form $\boldsymbol{\omega}$ on M one can assign the corresponding *vertical* flow h_t^+ on M, i.e., the flow at unit speed along the leaves of the foliation $\Re(\boldsymbol{\omega}) = 0$. The vertical flow preserves the measure $\mathfrak{m} = i(\boldsymbol{\omega} \wedge \overline{\boldsymbol{\omega}})/2$, the area form induced by $\boldsymbol{\omega}$. By a theorem of Katok [24], the flow h_t^+ is never mixing. The moduli space of abelian differentials carries a natural volume measure, called the Masur-Veech measure [25], [32]. For almost every Abelian differential with respect to the Masur-Veech measure, Masur [25] and Veech [32] independently and simultaneously proved that the flow h_t^+ is uniquely ergodic. Under additional assumptions on the combinatorics of the abelian differentials, weak mixing for almost all translation flows has been established by Veech in [33] and in full generality by Avila and Forni [3]. The spectrum of translation flows is therefore almost surely continuous and always has a singular component. No quantitative results have, however, previously been obtained about the spectral measure. Sinai [personal communication] posed the following

Problem. *Find the local asymptotics for the spectral measure of translation flows.*

1.2. **Formulation of the main result.** The aim of this paper is to give first quantitative estimates on the spectrum of translation flows. Let $\mathcal{H}(2)$ be the moduli space of abelian differentials on a surface of genus 2 with one zero of order two. The natural smooth Masur-Veech measure on the stratum $\mathcal{H}(2)$ is denoted by μ_2. Our main result is that for almost all abelian differentials in

$\mathcal{H}(2)$, the spectral measures of Lipschitz functions with respect to the corresponding translation flows have the Hölder property. Recall that for a square-integrable test function f, the spectral measure σ_f is defined by

$$\widehat{\sigma}_f(-t) = \langle f \circ h_t^+, f \rangle, \quad t \in \mathbb{R},$$

see Section 3.3 for details. A point mass for the spectral measure corresponds to an eigenvalue, so Hölder estimates for spectral measures quantify weak mixing for our systems.

Theorem 1.1. *There exists $\gamma > 0$ such that for μ_2-almost every abelian differential $(M, \boldsymbol{\omega}) \in \mathcal{H}(2)$ the following holds. For any $B > 1$ there exist constants $C = C(\boldsymbol{\omega}, B)$ and $r_0 = r_0(\boldsymbol{\omega}, B)$ such that for any Lipschitz function f on M, for all $x \in [B^{-1}, B]$ and $r \in (0, r_0)$ we have*

$$\sigma_f([x - r, x + r]) \le C \|f\|_L \cdot r^{\gamma}.$$

The proof uses the symbolic formalism of [9], namely, the representation of translation flows by flows along orbits of the asymptotic equivalence relation of a Markov compactum. The Hölder property is then reformulated as a consequence of a statement on Diophantine approximation involving the incidence matrices of the graphs forming the Markov compactum that codes the translation flow. These incidence matrices are, in turn, realizations of a renormalization cocycle, isomorphic, under our symbolic coding, to the Kontsevich-Zorich cocycle over the Teichmüller flow on the moduli space of abelian differentials. By the Oseledets multiplicative ergodic theorem, the cocycle admits a decomposition into Oseledets subspaces corresponding to the distinct Lyapunov exponents. Our argument in this paper requires that the Kontsevich-Zorich cocycle admit exactly two positive Lyapunov exponents and not have zero Lyapunov exponents; this is the reason why our main result only covers the stratum $\mathcal{H}(2)$.

We stress that our proof only works for the Masur-Veech smooth measure. This can informally be explained as follows. A translation flow can be represented as a suspension flow over an interval exchange transformation with a piecewise constant roof function. Take a measure on a stratum, invariant under the action of the Teichmüller flow. If one fixes an interval exchange transformation, then the invariant measure yields a conditional measure on the polyhedron of admissible tuples of heights of the rectangles. Our argument in its present form requires that this measure be absolutely continuous with respect to the natural Lebesgue measure; this property only holds for the Masur-Veech smooth measure.

The Hölder exponent $\gamma > 0$ obtained in Theorem 1.1 can be given explicitly, but it is very small and certainly not sharp, so we do not pursue this. It is an interesting question to determine the sharp exponent. More can be said about the Hölder exponent at $x = 0$ (assuming the test function f has zero mean), see Remark 3.2. Local estimates for spectral measures are obtained via growth estimates of twisted Birkhoff integrals.

The paper is organized as follows. In Section 2, we introduce the necessary setup of Markov compacta and Bratteli-Vershik (BV) transformations, as well the alternative, closely related framework, based on sequences of substitutions, or S-adic systems (see e.g. [36, 37, 38, 30, 9, 7] for futher background). We will be working in this "symbolic" framework for most of the paper, only returning to translation flows in the last Section 11. Estimates of twisted Birkhoff integrals in terms of matrix product are considered in Section 3, which builds on our paper [12]. The difference is that [12] was concerned with a single substitution, or equivalently, with a stationary Bratteli-Vershik diagram. In Section 4 we state the main theorem for *random* BV-systems, satisfying a certain list of conditions, among which the key condition is a uniform large deviations estimate. That theorem is proved in Sections 5-10. It should be emphasized that there are substantial technical difficulties in the transition from the stationary framework of [12] to the non-stationary setting of this paper. In the case of a single substitution matrix we could rely on estimates of the Vandermonde matrix, its determinant and its inverse. In this paper, we need similar estimates for the cocycle matrices. The Oseledets Theorem controls norms of these matrices and angles between different subspaces only up to subexponential errors. It is precisely in order to control these errors that we need the assumption that only two Lyapunov exponents be positive. In Section 10 we use a variant of the "Erdős-Kahane argument" that had originated in the theory of Bernoulli convolution measures, see [14, 23]. Finally, in Section 11 we conclude the proof by explaining how the symbolic coding of translation flows gives rise to suspension flows over BV-maps and by checking all the conditions required. The key probabilistic condition is derived from a (slight generalization of) large deviation estimate for the Teichmüller flow in [8].

1.3. **General remarks.** For comparison, consider the work on weak mixing by Avila-Forni [3] and the recent Avila-Delecroix [2]. Let us parametrize all possible eigenvalues by the line in $H^1(M; \mathbb{R})$ through the imaginary part of the 1-form $\boldsymbol{\omega}$ (we borrow here some of the wording from the introduction of [2], and also refer to that paper for the explanation of the terms used in this paragraph). The Veech criterion [33] says that an eigenvalue is parametrized by an element of "weak-stable lamination" associated to an acceleration of the Kontsevich-Zorich cocycle acting modulo $H^1(M, \mathbb{Z})$. In [3], a probabilistic method is applied to exclude non-trivial intersections of an arbitrary fixed line in $H^1(M; \mathbb{R})$ with the weak stable foliation to prove weak mixing for almost every interval exchange, and a simpler, "linear exclusion" is used to exclude such intersections and prove weak mixing for almost every translation flow. (On the other hand, [2] uses additional structure to prove weak-mixing of the translation flow in almost every direction for a given non-arithmetic Veech surface.) In order to prove Hölder continuity, roughly speaking, we need that there is a positive density of iterates that fall **outside** of a fixed neighborhood of $H^1(M, \mathbb{Z})$ along the orbit of the acceleration of the Kontsevich-Zorich cocycle, uniformly for a fixed line in $H^1(M; \mathbb{R})$. This requires much more delicate estimates, see Sections 9 and 10, hence more limited scope of our results.

Our inspiration came from the work of Salem, Erdős, and Kahane on Diophantine approximation and Bernoulli convolutions. A well-known problem, open since the 1930's, asks whether

$$(1.1) \qquad \{\lambda > 1 : \ \exists \, \alpha > 0 \ \text{ such that } \quad \lim_{n \to \infty} \|\alpha \lambda^n\|_{\mathbb{R}/\mathbb{Z}} = 0\}$$

is the set of PV-numbers, that is, λ an algebraic integer all of whose conjugates are less than one in absolute value. Salem [27] showed that the set in (1.1) is countable. In short, if

$$(1.2) \qquad \alpha \lambda^n = K_n + \varepsilon_n,$$

with K_n being the nearest integer, is such that $|\varepsilon_{n+j}|$ for $j \geq 0$, is sufficiently small, then all K_{n+j}, $j \geq 2$, are uniquely determined by K_n, K_{n+1}, and λ may be recovered as $\lim_{j \to \infty} \frac{K_{n+j}}{K_{n+j-1}}$, hence there are only countable many possibilities for λ. This may be compared with the "linear exclusion" from [3] (in the sense that in both cases the question of (non)convergence is addressed; the method is actually different).

The Bernoulli convolution measure ν_λ can be defined by the formula for its Fourier transform:

$$\widehat{\nu}_\lambda(t) = \prod_{n=0}^{\infty} \cos(2\pi \lambda^{-n} t), \quad t \in \mathbb{R}.$$

Erdős [13] observed that $\lim_{t \to \infty} |\widehat{\nu}_\lambda(t)| \neq 0$ for PV-numbers λ, which implies that the corresponding Bernoulli convolution measure ν_λ is singular. Salem (see [28]) proved that $\lim_{t \to \infty} |\widehat{\nu}_\lambda(t)| = 0$ in all other cases, but it is an open problem whether such ν_λ is necessarily absolutely continuous (see [21, 29, 31] for the recent progress in this direction). However, in another early paper, Erdős [14] proved that for almost all λ, $\widehat{\nu}_\lambda(t) = O(|t|^{-C(\lambda)})$, as $|t| \to \infty$, with $C(\lambda) \to \infty$ as $\lambda \to 1$, which implies absolute continuity with arbitrarily high smoothness for λ sufficiently close to one (Kahane [23] later showed that in the argument of Erdős one can replace "almost all" by "all outside of a set of any fixed positive Hausdorff dimension"). The argument of Erdős and Kahane builds on the Salem idea: if all ε_n in (1.2), outside a set of n's of small positive density are sufficiently small in absolute value, then "most" of the K_n's are uniquely determined by the previous ones, and we get a good covering of the "bad set" to yield the dimension estimate. A variation on this idea, which came to be called the "Erdős-Kahane argument", was used in [12], and a much more delicate version of it is developed in this work.

2. Preliminaries

2.1. **Markov compacta and Bratteli-Vershik transformations.** The reader is referred to [36, 37, 38, 30, 9] for further background.

Let $\mathfrak{G}$ be the set of all oriented graphs on m vertices such that there is an edge starting at every vertex and an edge ending at every vertex (we allow loops and multiple edges). For an edge $\mathfrak{e}$ we denote by $I(\mathfrak{e})$ and $F(\mathfrak{e})$ the initial and final vertices of $\mathfrak{e}$ correspondingly. Assume we are given a sequence $\{\Gamma_n\}_{n \in \mathbb{Z}}$ of graphs belonging to $\mathfrak{G}$. To this sequence we associate the *Markov compactum* of paths in our sequence of graphs:

$$(2.1) \qquad X = \{\overline{\mathfrak{e}} = (\mathfrak{e}_n)_{n \in \mathbb{Z}} : \ \mathfrak{e}_n \in \mathcal{E}(\Gamma_n), \ F(\mathfrak{e}_{n+1}) = I(\mathfrak{e}_n)\}.$$

We will also need the one-sided Markov compactum X_+ (respectively X_-), defined in the same way with elements $(\mathfrak{e}_n)_{n \geq 1}$ (respectively $(\mathfrak{e}_n)_{n \leq 0}$). A one-sided sequence of graphs in $\mathfrak{G}$ can also be considered as a *Bratteli diagram* of (finite) rank m. We then view its vertices as being arranged in levels $n \geq 0$, so that the graph Γ_n connects the vertices of level n to vertices of level $n - 1$. (Note that in some papers the direction of the edges is reversed.) Let $A_n(X) = A(\Gamma_n)$ be the incidence matrix of the graph Γ_n given by the formula

$$A_{ij}(\Gamma) = \#\{\mathfrak{e} \in \mathcal{E}(\Gamma) : \ I(\mathfrak{e}) = i, \ F(\mathfrak{e}) = j\}.$$

On the Markov compactum X we define the "future tail" and "past tail" equivalence relations, in which two infinite paths are equivalent iff they agree from some point on (into the future or into the past).

There is a standard construction of *telescoping* (= aggregation): for any sequence $1 = n_0 < n_1 < n_2 < \cdots$ we "concatenate" the graphs $\Gamma_{n_j}, \ldots, \Gamma_{n_{j+1}-1}$ to obtain $\Gamma'_j \in \mathfrak{G}$.

Standing Assumption. *The sequence Γ_n (after appropriate telescoping) contains infinitely many occurrences of a single graph Γ with a strictly positive incidence matrix, both in the past and in the future.*

In this case, as is well-known since the work of Furstenberg (see e.g. (16.13) in [20]), the Markov compactum X_+ is *uniquely ergodic*, which means that there is a unique invariant probability measure for the "future tail" equivalence relation. We need to develop this point in more detail. In fact, in this case we have (see [9, 1.9.5]) that there exist strictly positive vectors $\vec{\mathfrak{z}}^{(\ell)}, \vec{\mathfrak{u}}^{(\ell)}$, for $\ell \in \mathbb{Z}$, such that

$$\vec{\mathfrak{z}}^{(\ell)} = A_\ell^t \, \vec{\mathfrak{z}}^{(\ell+1)}, \quad \ell \in \mathbb{Z};$$

$$(2.2) \qquad \bigcap_{n\in\mathbb{N}} A_{\ell+1}^t \cdots A_{\ell+n}^t \mathbb{R}_+^m = \mathbb{R}_+ \, \vec{\mathfrak{z}}^{(\ell)}, \quad \ell \in \mathbb{Z};$$

$$\vec{\mathfrak{u}}^{(\ell)} = A_\ell \, \vec{\mathfrak{u}}^{(\ell-1)}, \quad \ell \in \mathbb{Z};$$

$$\bigcap_{n\in\mathbb{N}} A_{\ell-1} \cdots A_{\ell-n} \mathbb{R}_+^m = \mathbb{R}_+ \, \vec{\mathfrak{u}}^{(\ell)}, \quad \ell \in \mathbb{Z}.$$

The vectors are normalized by

$$|\vec{\mathfrak{z}}^{(0)}|_1 = 1, \ \ \langle \vec{\mathfrak{z}}^{(0)}, \vec{\mathfrak{u}}^{(0)} \rangle = 1.$$

As already mentioned, the Markov compactum X_+ is then uniquely ergodic, with the unique tail-invariant probability measure ν_+ given by

$$(2.3) \qquad \nu_+(X_j^+) = \mathfrak{z}_j^{(0)}, \quad \nu_+([\mathfrak{e}_1 \ldots \mathfrak{e}_\ell]) = \mathfrak{z}_{I(\mathfrak{e}_\ell)}^{(\ell)},$$

where X_j^+ is the set of one-sided paths $\mathfrak{e}_1 \mathfrak{e}_2 \ldots \in X_+$ such that $F(\mathfrak{e}_1) = j$ and $[\mathfrak{e}_1 \ldots \mathfrak{e}_\ell]$ is the cylinder set corresponding to the finite path. The advantage of working with 2-sided Bratteli diagrams, which is one of the key ideas of [9], is that one can similarly define the "dual" measure ν_- on the set of "negative paths" X_-, invariant under the "past tail" equivalence relation. Then $\nu = \nu_+ \times \nu_-$ is a probability measure on X.

Now suppose that a linear ordering (Vershik's ordering) is defined on the set $\{\mathfrak{e} \in \mathcal{E}(\Gamma_\ell) : I(\mathfrak{e}) = i\}$ for all $i \leq m$ and $\ell \in \mathbb{Z}$. This induces a partial lexicographic ordering $\mathfrak{o}$ on the Markov compactum X; more precisely, two paths are comparable if they agree for $n \geq t$ for some $t \in \mathbb{Z}$. (Also two paths in X_- are comparable if they end at the same vertex.) Restricting this ordering to the 1-sided compactum X_+, we obtain the *adic*, or *Bratteli-Vershik* (BV) transformation $\mathfrak{T}$, defined as the immediate successor of a path $\overline{\mathfrak{e}}$ in the ordering $\mathfrak{o}$. (See also the work of S. Ito [22], where a similar construction had appeared earlier.) Let $\mathrm{Max}(\mathfrak{o})$ be the set of paths in X_+ such that its every edge is maximal. It is easy to see that $\mathrm{card}(\mathrm{Max}(\mathfrak{o})) \leq m$. Similarly define the set of minimal paths $\mathrm{Min}(\mathfrak{o})$, and let $\widetilde{X}_+$ be the set of paths, which are not tail equivalent to any of the paths in $\mathrm{Min}(\mathfrak{o}) \cup \mathrm{Max}(\mathfrak{o})$. Then the $\mathbb{Z}$-action $\{\mathfrak{T}^n\}_{n\in\mathbb{Z}}$ is well-defined on $\widetilde{X}_+$. Since we are excluding a countable set, the action is defined almost everywhere with respect to any non-atomic measure; certainly, ν_+-a.e. in the uniquely ergodic case. We similarly define $\widetilde{X}$ as the set of bi-infinite paths in X which are not forward tail-equivalent to any of the maximal or minimal paths. Note that invariant measures for the future tail equivalence relation are precisely the invariant measures for the BV map, so we get unique ergodicity of the system $(X_+, \mathfrak{T})$ under our standing assumption. An easy, but important, fact is that the construction of telescoping/aggregation preserves the Vershik ordering, and the corresponding BV-systems are isomorphic.

We shall also consider suspension flows over BV-systems, with a piecewise-constant roof function depending only on the vertex at the level 0. More precisely, let X_+ be a one-sided Markov compactum with a Vershik ordering and BV-transformation $\mathfrak{T}$. For a strictly positive vector

$\vec{s} = (s_1, \ldots, s_m)$ define the roof function $\phi_{\vec{s}}$ to be equal to s_j on the cylinder set X_j^+. The resulting space will be denoted $\mathfrak{X}^{\vec{s}}$:

$$\mathfrak{X}^{\vec{s}} := \bigsqcup_{j=1}^{m} X_j^+ \times [0, s_j] \, / \! \sim, \quad \text{with} \ \ (\mathfrak{e}, \phi_{\vec{s}}(\mathfrak{e})) \sim (\mathfrak{T}\mathfrak{e}, 0),$$

on which we consider the usual suspension flow $\{h_t\}_{t \in \mathbb{R}}$. It preserves the measure induced by the $\mathfrak{T}$-invariant measure ν_+ on X_+ and the Lebesgue measure on $\mathbb{R}$. We will need the following

Lemma 2.1. [9, Section 2.5] *Given a Vershik ordering on a uniquely ergodic Markov compactum X with the unique invariant measure ν, there is a symbolic flow $(X, h_t^+)_{t \in \mathbb{R}}$, defined on $\widetilde{X}$, so ν-a.e. on X, which is measurably conjugate to the suspension flow $(\mathfrak{X}^{\vec{s}}, h_t)_{t \in \mathbb{R}}$ over $(X_+, \mathfrak{T})$, with $s_j = \mathfrak{u}_j^{(1)}$, $j \leq m$. Moreover , the conjugating map $\mathfrak{F} : X \to \mathfrak{X}^{\vec{s}}$ is given by*

$$\mathfrak{F}(\overline{\mathfrak{e}}) = (P_+ \overline{\mathfrak{e}}, t), \quad where \ \ t = \nu_-(\{\mathfrak{a} \in X_- : \ I(\mathfrak{a}_0) = F(\mathfrak{e}_1), \ \mathfrak{a} \preccurlyeq P_- \mathfrak{e}\}),$$

where P_+, P_- are the truncations from X to X_+, X_- respectively and $\preccurlyeq$ is the Vershik order. The map $\mathfrak{F}$ is well-defined on $\widetilde{X}$ and its inverse is well-defined over $\widetilde{X}_+$.

2.2. **Weakly Lipschitz functions.**

Following [10, 9], we consider the space of *weakly Lipschitz functions* on a uniquely ergodic Markov compactum X with the probability measure ν_+ invariant for the "forward tail" equivalence relation and a Vershik ordering $\mathfrak{o}$. Recall that $\widetilde{X}$ denotes the set of paths in x that are not (forward) tail equivalent to any of the maximal or minimal paths in the Vershik ordering. We say that f is weakly Lipschitz and write $f \in \mathrm{Lip}_w^+(X)$ if f is defined and bounded on $\widetilde{X}$, and there exists $C > 0$ such that for all $\overline{\mathfrak{e}}, \overline{\mathfrak{e}}' \in \widetilde{X}$, satisfying $\mathfrak{e}_k = \mathfrak{e}_k'$ for $-\infty < k \leq n$, with $n \in \mathbb{N}$, we have

$$(2.4) \qquad\qquad |f(\overline{\mathfrak{e}}) - f(\overline{\mathfrak{e}}')| \leq C\nu_+([\mathfrak{e}_1 \ldots \mathfrak{e}_n]).$$

The norm in $\mathrm{Lip}_w^+(X)$ is defined by

$$(2.5) \qquad\qquad \|f\|_L := \|f\|_{\sup} + \widetilde{C},$$

where $\widetilde{C}$ is the infimum of the constants in (2.4). Note that a weakly Lipschitz function is mapped into a weakly Lipschitz function when telescoping of the diagram is performed, and this does not increase the norm $\| \cdot \|_L$.

We analogously define the space $\mathrm{Lip}_w^+(\mathfrak{X}^{\vec{s}})$ of weakly Lipschitz functions on the space $\mathfrak{X}^{\vec{s}}$ of the suspension flow over $(X_+, \mathfrak{T})$ with the roof function determined by the vector $\vec{s} \in \mathbb{R}_+^m$. Namely, $f \in \mathrm{Lip}_w^+(\mathfrak{X}^{\vec{s}})$ if it is defined and bounded on $\widetilde{X}_+$ and there exists $C > 0$ such that for all $\overline{\mathfrak{e}}, \overline{\mathfrak{e}}' \in \widetilde{X}_+$, satisfying $\mathfrak{e}_k = \mathfrak{e}_k'$ for $k \leq n$, with $n \in \mathbb{N}$, and all $t \in [0, s_{F(\mathfrak{e}_1)}]$ we have

$$(2.6) \qquad\qquad |f(\overline{\mathfrak{e}}, t) - f(\overline{\mathfrak{e}}', t)| \leq C\nu_+([\mathfrak{e}_1 \ldots \mathfrak{e}_n]).$$

The norm $\|f\|_L$ is defined as in (2.5).

Note that weakly Lipschitz functions are not Lipschitz in the "transverse" direction, corresponding to the "past" in the 2-sided Markov compactum and to the vertical direction in the space of the suspension flow. Note also that if $f \in \mathrm{Lip}_w^+(X)$, then $f \circ \mathfrak{F}^{-1} \in \mathrm{Lip}_w^+(\mathfrak{X}^{\vec{s}})$, with the same norm, where $\mathfrak{F}$ is defined in Lemma 2.1.

2.3. Substitutions.

Along with the Markov compactum and BV-transformation, it is convenient to use the language of substitutions (see e.g. [17] for further background). Consider the alphabet $\mathcal{A} = \{1, \ldots, m\}$, which is identified with the vertex set of all the graphs Γ_n. A *substitution* is a map $\zeta : \mathcal{A} \to \mathcal{A}^+$, extended to $\mathcal{A}^+$ and $\mathcal{A}^{\mathbb{N}}$ by concatenation. Given a Vershik ordering $\mathfrak{o}$ on a 1-sided Bratteli diagram $\{\Gamma_j\}_{j \geq 1}$, the substitution ζ_j takes every $b \in \mathcal{A}$ into the word in $\mathcal{A}$ corresponding to all the vertices to which there is a Γ_j-edge starting at b, in the order determined by $\mathfrak{o}$. Formally, the *length* of the word $\zeta_j(b)$ is

$$|\zeta_j(b)| = \sum_{a=1}^m A_{b,a}(\Gamma_j),$$

and the substitution itself is given by

$$(2.7) \qquad \zeta_j(b) = u_1^{b,j} \ldots u_{|\zeta_j(b)|}^{b,j}, \quad b \in \mathcal{A}, \quad j \geq 0,$$

where $(b, u_i^{b,j}) \in \mathcal{E}(\Gamma_j)$, listed in the linear order prescribed by $\mathfrak{o}$. Substitutions, extended to $\mathcal{A}^+$, can be composed in the usual way as transformations $\mathcal{A}^+ \to \mathcal{A}^+$. We will use the notation

$$\zeta^{[n_1,n_2]} := \zeta_{n_1} \circ \cdots \circ \zeta_{n_2}, \quad n_2 \geq n_1,$$

and

$$\zeta^{[n]} := \zeta^{[1,n]}, \quad n \geq 1.$$

Given a substitution ζ, its *substitution matrix* is defined by

$$\mathsf{S}_\zeta(a,b) := \# \text{ symbols } a \text{ in } \zeta(b).$$

Observe that $\mathsf{S}_{\zeta_1 \circ \zeta_2} = \mathsf{S}_{\zeta_1} \mathsf{S}_{\zeta_2}$. We will denote $\mathsf{S}_n = \mathsf{S}_{\zeta_n}$. Notice that the transpose S_n^t is exactly the incidence matrix $A_n = A(\Gamma_n)$ by the definition of ζ_n:

$$(2.8) \qquad \mathsf{S}_{\zeta_n}^t = A(\Gamma_n).$$

We will use the notation

$$\mathsf{S}^{[n_1,n_2]} := \mathsf{S}_{\zeta^{[n_1,n_2]}} \quad \text{and} \quad \mathsf{S}^{[n]} := \mathsf{S}_{\zeta^{[n]}}.$$

Next, we associate to any $\overline{\mathfrak{e}} \in X_+$, its "horizontal sequence" in the alphabet $\mathcal{A}$, defined by

$$\mathfrak{h}(\overline{\mathfrak{e}}) := x = (x_n)_n, \quad \text{where} \quad x_n = x_n(\overline{\mathfrak{e}}) = b \quad \text{whenever} \quad F(\mathfrak{T}^n(\overline{\mathfrak{e}})_1) = b, \quad n \in \mathbb{Z},$$

that is, we keep track of the vertex at level zero, applying the BV-transformation. The horizontal sequence is just the symbolic dynamics of $\mathfrak{T}$ with respect to the 0-level cylinder partition. We get a full 2-sided infinite sequence $\mathfrak{h}(\overline{\mathfrak{e}}) = x = (x_n)_{n \in \mathbb{Z}}$ whenever the (2-sided) orbit of $\overline{\mathfrak{e}}$ under $\mathfrak{T}$ does

not hit a maximal or a minimal path. (By our assumptions, no orbit of $\mathfrak{T}$ can hit both.) Thus $\mathfrak{h} : \widetilde{X}_+ \to \mathcal{A}^{\mathbb{Z}}$ is well-defined. We obtain, by definition, the following commutative diagram:

$$
\begin{array}{ccc}
\widetilde{X}_+ & \xrightarrow{\;\mathfrak{T}\;} & \widetilde{X}_+ \\
{\scriptstyle \mathfrak{h}}\downarrow & & \downarrow{\scriptstyle \mathfrak{h}} \\
\mathcal{A}^{\mathbb{Z}} & \xrightarrow{\;T\;} & \mathcal{A}^{\mathbb{Z}}
\end{array}
$$

where T is the left shift on $\mathcal{A}^{\mathbb{Z}}$. Of course, the map $\mathfrak{h}$ is far from being surjective. In order to understand its image, it is useful to have a more explicit algorithm for $\mathfrak{h}(\mathfrak{e})$. Suppose that the path $\mathfrak{e} \in X_+$ goes through the vertices $b_0, b_1, \ldots$, that is, $b_n = F(\mathfrak{e}_{n+1})$. Recalling the definition of the substitutions ζ_n we can write

$$(2.9) \qquad\qquad \zeta_n(b_n) = u_{n-1} b_{n-1} v_{n-1}, \quad n \geq 1,$$

where u_{n-1} and v_{n-1} are words, possibly empty. Note that there may be more than one occurrence of b_{n-1} in $\zeta_n(b_n)$, but we choose the representation (2.9) according to the edge $\mathfrak{e}_n$. Consider the following sequence of words U_n, $n \geq 0$, defined inductively. We start with

$$U_0 = u_0 . b_0 v_0,$$

where the dot $.$ separates negative and positive coordinates. Then U_{n+1} is obtained from U_n inductively, by appending $\zeta_{n+1}(u_{n+1})$ from the left and $\zeta_{n+1}(v_{n+1})$ from the right. If we disregard the location of the dot, we simply have

$$U_n = \zeta_1 \circ \cdots \circ \zeta_{n+1}(b_{n+1}) = \zeta^{[n+1]}(b_{n+1}), \quad n \geq 0.$$

When we take the location of the dot into account, typically, the words U_n will "grow" to infinity, both left and right, to a limiting 2-sided sequence, which is exactly $\mathfrak{h}(\mathfrak{e})$:

$$(2.10) \qquad\qquad \mathfrak{h}(\mathfrak{e}) = \ldots \zeta_2(u_2) \zeta_1(u_1) u_0 . b_0 v_0 \zeta_1(v_1) \zeta_2(v_2) \ldots$$

The other alternative is that it grows to infinity only on one side, which happens if and only if $\mathfrak{e}$ is tail equivalent to either minimal or a maximal path. Denote

$$\mathcal{Y} := \mathrm{clos}(\mathfrak{h}(\widetilde{X}_+)),$$

where "clos" denotes the closure in the product topology. Now the following is clear.

Lemma 2.2. *The space $\mathcal{Y} \subset \mathcal{A}^{\mathbb{Z}}$ is exactly the set of 2-sided sequences x with the property that any subword of x appears as a subword of $\zeta^{[n]}(b)$ for some $b \in \mathcal{A}$ and $n \geq 1$.*

Remark. Dynamical systems $(\mathcal{Y}, T)$ have been studied under the name of *S-adic systems*. They were originally introduced by S. Ferenczi [15], with the additional assumption that there are finitely many different substitutions in the sequence $\{\zeta_j\}$; however, more recently this restriction has been removed, see e.g. [7].

Let $h_i^{(n)}$ be the number of finite paths $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$ such that $I(\mathfrak{e}_n) = i$. (This is the height of the Rokhlin tower for the BV-map.) We get a sequence of real vectors $\vec{h}^{(n)}$ which satisfy the equations:

$$(2.11) \qquad \vec{h}^{(n+1)} = A_{n+1}\vec{h}^{(n)} = \mathsf{S}_{n+1}^t \vec{h}^{(n)}, \quad h_i^{(n)} = |\zeta^{[n]}(i)|, \quad n \geq 1.$$

Let $\mathfrak{s}$ be the left shift transformation on the 1-sided Markov compactum: $\mathfrak{s}(\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3, \ldots) = (\mathfrak{e}_2, \mathfrak{e}_3, \ldots)$. We thus obtain a sequence of Markov compacta $X_+^{(\ell)}$ for $\ell \geq 0$, with $X_+^{(0)} := X_+$, so that $\mathfrak{s} : X_+^{(\ell)} \to X_+^{(\ell+1)}$ for all ℓ. The Vershik ordering and BV-transformation are naturally induced on the whole family. We can then consider the horizontal sequence map

$$\mathfrak{h} : \widetilde{X}_+^{(\ell)} \to \mathcal{A}^{\mathbb{Z}}.$$

Its image, denoted by $\mathcal{Y}^{(\ell)}$, is described similarly to $\mathcal{Y} = \mathcal{Y}^{(0)}$, as the set of all sequences in $\mathcal{A}^{\mathbb{Z}}$ whose every subword occurs as a block in $\zeta^{[\ell+1,n]}(b)$ for some $n \geq \ell + 1$ and $b \in \mathcal{A}$.

A substitution ζ acts on $\mathcal{A}^{\mathbb{Z}}$ as follows:

$$\zeta(\ldots a_{-1}a_0.a_1 \ldots) = \ldots \zeta(a_{-1})\zeta(a_0).\zeta(a_1) \ldots$$

Definitions imply that we have a sequence of surjective maps $\zeta_\ell : \mathcal{Y}^{(\ell)} \to \mathcal{Y}^{(\ell-1)}, \quad \ell \geq 1$. It follows from definitions (and the explicit formulas for $\mathfrak{h}$) that

$$x = T^{k-1}\zeta_1(x'),$$

where k is the number (rank) of the edge $\mathfrak{e}_0$ in the Vershik ordering. Of course, similar formulas relate $\mathfrak{h}(\mathfrak{e})$ and $\mathfrak{h}(\mathfrak{s}\mathfrak{e})$ for $\mathfrak{e} \in X_+^{(\ell)}$. (Recall that T denotes the left shift on $\mathcal{A}^{\mathbb{Z}}$.)

3. Estimating the growth of exponential sums and matrix products

We use the following convention for the Fourier transform of functions and measures: given $\psi \in L^1(\mathbb{R})$ we set $\widehat{\psi}(t) = \int_{\mathbb{R}} e^{-2\pi i \omega t} \psi(\omega) \, d\omega$, and for a probability measure ν on $\mathbb{R}$ we let $\widehat{\nu}(t) = \int_{\mathbb{R}} e^{-2\pi i \omega t} \, d\nu(\omega)$.

3.1. Spectral measures and twisted Birkhoff integrals.

Since our goal is to obtain estimates of spectral measures, we recall how they are defined for flows. Given a measure-preserving flow $(Y, h_t, \mu)_{t \in \mathbb{R}}$ and a test function $f \in L^2(Y, \mu)$, there is a finite positive Borel measure σ_f on $\mathbb{R}$ such that

$$\widehat{\sigma}_f(-\tau) = \int_{-\infty}^{\infty} e^{2\pi i \omega \tau} \, d\sigma_f(\omega) = \langle f \circ h_\tau, f \rangle \quad \text{for } \tau \in \mathbb{R}.$$

In order to obtain local bounds on the spectral measure, we can use growth estimates of the "twisted Birkhoff integral"

$$(3.1) \qquad S_R^{(y)}(f, \omega) := \int_0^R e^{-2\pi i \omega \tau} f \circ h_\tau(y) \, d\tau.$$

The following lemma is standard; a proof may be found in [12, Lemma 4.3].

Lemma 3.1. *Suppose that for some fixed $\omega \in \mathbb{R}$, $R_0 > 0$, and $\alpha \in (0,1)$ we have*

$$(3.2) \qquad \left\| S_R^{(y)}(f,\omega) \right\|_{L^2(Y,\mu)} \le C_1 R^\alpha \quad \text{for all } R \ge R_0.$$

Then

$$(3.3) \qquad \sigma_f([\omega - r, \omega + r]) \le \pi^2 2^{-2\alpha} C_1^2 r^{2(1-\alpha)} \quad \text{for all } r \le (2R_0)^{-1}.$$

Remark 3.2. 1. Since (Y,μ) is a probability space, the L^2-estimate (3.2) obviously follows from a uniform estimate. We only need L^2 estimates for the proof of our main result. Nonetheless, we expect that the uniform estimates of this paper would have further applications.

2. Estimates of twisted Birkhoff sums have been used for a number of different dynamical systems recently; in particular, see the work of Forni and Ulcigrai [19] on the Lebesgue spectrum for smooth time changes of the horocycle flow.

3. For $\omega = 0$ the expression (3.1) reduces to the usual Birkhoff integral, for which sharp estimates and asymptotics are known (under the assumption that the test function f has zero mean) in a number of cases. It should be possible to obtain precise asymptotics of the spectral measure at zero for almost every translation flow in the context of Theorem 1.1, even for a general stratum in genus $g > 1$. We expect that it is governed by the second Lyapunov exponent, based on the results of [10], analogously to [12, Theorem 6.2]

4. It is easy to see that $\left\| S_R^{(y)}(f,\omega) \right\|_{L^2(Y,\mu)} = O(R)$, and $o(R)$ indicates the absence of an eigenvalue at ω. The exponent $\alpha = 1/2$ in (3.2) corresponds, in some sense, to the Lebesgue spectrum (this is made precise at zero for self-similar translation flows with periodic renormalization in [12, Theorem 6.2]). Whereas $\alpha < 1/2$ is possible at some points, for σ_f-a.e. ω, the infimum of α, for which (3.2) holds, is at least $1/2$. Indeed, (3.3) implies that $\underline{d}(\sigma_f, \omega) = \liminf_{r \to 0} \frac{\log \sigma_f(B(\omega,r))}{\log r} \ge 2(1-\alpha)$, and it is well-known that

$$\dim_H(\sigma_f) = \sup\{s : \underline{d}(\sigma_f, \omega) \ge s \text{ for } \sigma_f\text{-a.e. } \omega\} \le 1,$$

see e.g. [16, Prop. 10.2].

3.2. **Exponential sums corresponding to suspension flows.** Let X_+ be a one-sided Markov compactum with a Vershik ordering and BV-transformation $\mathfrak{T}$. For a strictly positive vector $\vec{s} = (s_1, \ldots, s_m)$ we define the roof function $\phi_{\vec{s}}$ to be equal to s_a on the cylinder set X_a^+, as in Section 2, and obtain the suspension flow $(\mathfrak{X}^{\vec{s}}, h_t)$.

Recall that for $\overline{\mathfrak{e}} \in X_+$ (minus a countable exceptional set) we defined its horizontal sequence $\mathfrak{h}(\overline{\mathfrak{e}}) = (x_n)_{n \in \mathbb{Z}}$, in such a way that the BV-transformation intertwines the left shift. Similarly, we can associate to $(\overline{\mathfrak{e}}, t) \in \mathfrak{X}^{\vec{s}}$ a tiling of the line $\mathbb{R}$: a symbol a corresponds to a closed line segment of length s_a (labeled by a), and these line segments are "strung together" according to the symbolic sequence $\mathfrak{h}(\overline{\mathfrak{e}})$. The tile corresponding to x_0 should contain the origin at the distance

t from the left endpoint. This defines a map $\widetilde{\mathfrak{h}}$ from $\mathfrak{X}^{\vec{s}}$ to a "tiling space," which intertwines the flow h_τ and the left shift by τ.

Our goal is to obtain growth estimates for twisted Birkhoff integrals (3.1). We start with test functions depending only on the cylinder set X_a and the height t. More precisely, given some functions $\psi_a \in C([0, s_a])$, $a \in \mathcal{A}$, let

$$f = \sum_{a \in \mathcal{A}} c_a f_a, \quad \text{with} \ \ f_a(\overline{\mathfrak{e}}, t) = \mathbb{1}_{\mathfrak{X}_a} \psi_a(t), \ \ \text{where} \ \ \mathfrak{X}_a = X_a \times [0, s_a].$$

For a word v in the alphabet $\mathcal{A}$ denote by $\vec{\ell}(v) \in \mathbb{Z}^m$ its "population vector" whose j-th entry is the number of j's in v, for $j \le m$. We will need the "tiling length" of v defined by

$$(3.4) \qquad\qquad\qquad\qquad |v|_{\vec{s}} := \langle \vec{\ell}(v), \vec{s} \rangle.$$

The following property will be used frequently: for an arbitrary substitution ξ, $\vec{s} > 0$, and word $U \in \mathcal{A}^+$ we have

$$(3.5) \qquad\qquad |U|_{\mathsf{S}^t_\xi \vec{s}} = \langle \vec{\ell}(U), \mathsf{S}^t_\xi \vec{s} \rangle = \langle \mathsf{S}_\xi \vec{\ell}(U), \vec{s} \rangle = \langle \vec{\ell}(\xi(U)), \vec{s} \rangle = |\xi(U)|_{\vec{s}}.$$

For $v = v_0 \ldots v_{N-1} \in \mathcal{A}^+$

$$(3.6) \qquad\qquad \Phi_a^{\vec{s}}(v, \omega) = \sum_{j=0}^{N-1} \delta_{v_j, a} \exp(-2\pi i \omega |v_0 \ldots v_{j-1}|_{\vec{s}}),$$

where the term for $j = 0$ is equal to one by convention. Then a straightforward calculation shows

$$(3.7) \qquad\quad S_R^{(\overline{\mathfrak{e}}, 0)}(f_a, \omega) = \widehat{\psi}_a(\omega) \cdot \Phi_a^{\vec{s}}(x[0, N-1], \omega) \quad \text{for} \ \ R = |x[0, N-1]|_{\vec{s}},$$

where $(x_n)_{n \in \mathbb{Z}} = \mathfrak{h}(\overline{\mathfrak{e}})$. Moreover, the horizontal sequence can be represented as a concatenation of long blocks of the form $\zeta^{[n]}(b)$, $b \in \mathcal{A}$ (not necessarily starting at the 0-th position); therefore, estimates of twisted Birkhoff sums for an arbitrary sequence may be reduced to those for $\zeta^{[n]}(b)$, and for the latter the renormalization naturally leads to matrix products. This is what we do next.

3.3. Setting up matrix products. Observe that for any two words u, v and the concatenated word uv we have

$$(3.8) \qquad\qquad \Phi_a^{\vec{s}}(uv, \omega) = \Phi_a^{\vec{s}}(u, \omega) + e^{-2\pi i \omega |u|_{\vec{s}}} \cdot \Phi_a^{\vec{s}}(v, \omega).$$

Recalling (2.7), we can write

$$\zeta^{[n]}(b) = \zeta^{[n-1]}(\zeta_n(b)) = \zeta^{[n-1]}(u_1^{b,n}) \ldots \zeta^{[n-1]}(u_{|\zeta_n(b)|}^{b,n}), \ n \ge 1,$$

where we use the convention $\zeta^{[0]} := Id$. Hence (3.8) implies for all $a, b \in \mathcal{A}$:

$$\Phi_a^{\vec{s}}(\zeta^{[n]}(b), \omega) = \sum_{j=1}^{|\zeta_n(b)|} \exp\left[-2\pi i \omega \left(|\zeta^{[n-1]}(u_1^{b,n} \ldots u_{j-1}^{b,n})|_{\vec{s}}\right)\right] \Phi_a^{\vec{s}}(\zeta^{[n-1]}(u_j^{b,n}), \omega), \ n \ge 1,$$

(for $j = 1$, the exponential reduces to $\exp(0) = 1$ by definition). For $n \geq 0$ consider the $m \times m$ matrix-function $\mathbf{\Pi}_n(\omega)$ defined by

$$(3.9) \qquad \mathbf{\Pi}_n^{\vec{s}}(\omega) = \begin{pmatrix} \Phi_1^{\vec{s}}(\zeta^{[n]}(1), \omega) & \dots & \Phi_m^{\vec{s}}(\zeta^{[n]}(1), \omega) \\ \dots\dots\dots & \dots & \dots\dots\dots \\ \Phi_1^{\vec{s}}(\zeta^{[n]}(m), \omega) & \dots & \Phi_m^{\vec{s}}(\zeta^{[n]}(m), \omega). \end{pmatrix}.$$

It follows that

$$(3.10) \qquad \mathbf{\Pi}_n^{\vec{s}}(\omega) = \mathbf{M}_n^{\vec{s}}(\omega)\mathbf{\Pi}_{n-1}^{\vec{s}}(\omega), \quad n \geq 1,$$

where $\mathbf{M}_n^{\vec{s}}(\omega)$ is an $m \times m$ matrix-function, whose matrix elements are trigonometric polynomials given by

$$(3.11) \qquad (\mathbf{M}_n^{\vec{s}}(\omega))(b, c) = \sum_{j \leq |\zeta_n(b)|: \ u_j^{b,n} = c} \exp\left[-2\pi i \omega \left(|\zeta^{[n-1]}(u_1^{b,n} \dots u_{j-1}^{b,n})|_{\vec{s}} \right) \right], \quad n \geq 1.$$

Note that $\mathbf{M}_n^{\vec{s}}(0) = \mathsf{S}_n^t$, the transpose of the n-th substitution matrix, for all $n \geq 1$. Since $\mathbf{\Pi}_0^{\vec{s}}(\omega) = I$ (the identity matrix), it follows from (3.10) that

$$(3.12) \qquad \mathbf{\Pi}_n^{\vec{s}}(\omega) = \mathbf{M}_n^{\vec{s}}(\omega)\mathbf{M}_{n-1}^{\vec{s}}(\omega) \cdots \mathbf{M}_1^{\vec{s}}(\omega).$$

There is another way to express the matrices $\mathbf{M}_n^{\vec{s}}(\omega)$ which will be useful below. It follows from (3.6), (3.11), and (3.5) that

$$\mathbf{M}_n^{\vec{s}}(\omega)(b, c) = (\Phi_c)^{\mathsf{S}_{\zeta^{[n-1]}}^t \vec{s}}(\zeta_n(b), \omega).$$

This motivates the following definition for two arbitrary substitutions ξ_1, ξ_2:

$$(3.13) \qquad \mathbf{M}_{\xi_1, \xi_2}^{\vec{s}}(\omega)(b, c) = (\Phi_c)^{\mathsf{S}_{\xi_1}^t \vec{s}}(\xi_2(b), \omega),$$

so that

$$(3.14) \qquad \mathbf{M}_n^{\vec{s}} = \mathbf{M}_{\zeta^{[n-1]}, \zeta_n}^{\vec{s}}.$$

A straightforward verification yields the following identity for arbitrary substitutions ξ_1, ξ_2, ξ_3:

$$(3.15) \qquad \mathbf{M}_{\xi_1, \xi_2 \xi_3}^{\vec{s}} = \mathbf{M}_{\xi_1 \xi_2, \xi_3}^{\vec{s}} \cdot \mathbf{M}_{\xi_1, \xi_2}^{\vec{s}}.$$

3.4. Estimating matrix products.

Definition 3.3. *A word v is called a* **good return word** *for the substitution ζ if v starts with some symbol c (which can be any element of $\mathcal{A}$) and vc occurs in the word $\zeta(b)$ for every $b \in \mathcal{A}$. Denote by $GR(\zeta)$ the set of good return words for ζ.*

One of our main assumptions will be that a fixed substitution ζ, with a strictly positive matrix $\mathsf{S}_\zeta =: Q$ and nonempty set of good return words, appears infinitely often in the sequence ζ_j. Thus, it is convenient to perform "telescoping" of the Bratteli-Vershik diagram and assume from the start that every substitution ζ_j has the form

$$\zeta_j = \zeta \xi_j \zeta, \tag{3.16}$$

where ξ_j is an arbitrary substitution.

Denote by $\vec{1}$ the vector of all 1's, and let $\|x\|_{\mathbb{R}/\mathbb{Z}}$ be the distance from $x \in \mathbb{R}$ to the nearest integer. For a strictly positive square $m \times n$ matrix A let

$$\mathrm{col}(A) = \max_{i,j,k} \frac{A_{ij}}{A_{kj}}.$$

It is well-known and easy to check that if Q is strictly positive $m \times m$ matrix and A is any non-negative $m \times n$ matrix, then we have

$$\mathrm{col}(QA) \leq \mathrm{col}(Q). \tag{3.17}$$

Proposition 3.4. *Let X_+ be a one-sided Markov compactum with a Vershik ordering, and let ζ_j be the corresponding sequence of substitutions, given by (2.7). Suppose that there exists a substitution ζ with a nonempty set of good return words, such that $Q = \mathsf{S}_\zeta$ is strictly positive and $\zeta_j = \zeta \xi_j \zeta$ for some substitution ξ_j for all $j \geq 1$. Then there exists $c_1 \in (0,1)$, depending only on the substitution ζ, such that for all $a, b \in \mathcal{A}$ and $N \in \mathbb{N}$, $\omega \in [0,1)$, and $\vec{s} > \vec{0}$,*

$$|\Phi_a^{\vec{s}}(\zeta^{[N]}(b), \omega)| \leq \|\mathsf{S}^{[N]}\|_1 \prod_{n \leq N-1} \left(1 - c_1 \cdot \max_{v \in GR(\zeta)} \|\omega|\zeta^{[n]}(v)|_{\vec{s}}\|_{\mathbb{R}/\mathbb{Z}}^2 \right). \tag{3.18}$$

In fact, we can take

$$c_1 = \left(2m \cdot \max_{i,j} Q_{ij} \cdot \mathrm{col}(Q^t) \right)^{-1}. \tag{3.19}$$

Proof. This is similar to the proof of [12, Proposition 3.2], but there are a number of new technical details. Let $\vec{e}_a$ denote the unit basis vector in $\mathbb{R}^m$ corresponding to $a \in \mathcal{A}$. In view of (3.9), it suffices to estimate $\mathbf{\Pi}_N^{\vec{s}}(\omega)\vec{e}_a$. We will use the following notation:

- for vectors $\vec{x}, \vec{y} \in \mathbb{R}^m$, the inequality $\vec{x} \leq \vec{y}$ means componentwise inequality, and similarly for real-valued matrices;
- the operation of taking absolute values of all entries for a vector $\vec{x}$ and a matrix A will be denoted $\vec{x}^{|\cdot|}$ and $A^{|\cdot|}$.

It is clear that for any, generally speaking, rectangular matrices A, B such that the product AB is well-defined, we have

$$(AB)^{|\cdot|} \leq A^{|\cdot|} B^{|\cdot|}. \tag{3.20}$$

We fix ω and $\vec{s}$ and omit them from notation, so that $\mathbf{M}_n \equiv \mathbf{M}_n^{\vec{s}}(\omega)$ and $\mathbf{\Pi}_n \equiv \mathbf{\Pi}_n^{\vec{s}}(\omega)$. Observe that (3.20) and (3.10) imply for all $n \in \mathbb{N}$:

$$(3.21) \qquad (\mathbf{\Pi}_n \vec{e}_a)^{|\cdot|} = (\mathbf{M}_n \cdots \mathbf{M}_1 \vec{e}_a)^{|\cdot|} \leq \mathbf{M}_n^{|\cdot|} \cdots \mathbf{M}_1^{|\cdot|} \vec{e}_a \leq \mathbf{M}_n^{|\cdot|} \cdots \mathbf{M}_1^{|\cdot|} \vec{1}.$$

In view of (3.14), (3.15), and (3.16), we have

$$
\begin{aligned}
\mathbf{M}_n = \mathbf{M}_{\zeta[n-1],\zeta_n} \;\; &= \;\; \mathbf{M}_{\zeta[n-1],\zeta\xi_n\zeta} \\
&= \;\; \mathbf{M}_{\zeta[n-1]\zeta\xi_n,\zeta} \cdot \mathbf{M}_{\zeta[n-1],\zeta\xi_n} \\
(3.22) \qquad &= \;\; \mathbf{M}_{\zeta[n-1]\zeta\xi_n,\zeta} \cdot \mathbf{M}_{\zeta[n-1]\zeta,\xi_n} \cdot \mathbf{M}_{\zeta[n-1],\zeta}.
\end{aligned}
$$

By the definition of a good return word, for any $v \in GR(\zeta)$ and $b \in \mathcal{A}$, we can write

$$(3.23) \qquad \zeta(b) = p^{(b)} v c q^{(b)},$$

where $p^{(b)}$ and $q^{(b)}$ are words, possibly empty, and v starts with c. Let ξ be any substitution on $\mathcal{A}$. First we are going to estimate the absolute value of

$$\mathbf{M}_{\xi,\zeta}(b,c) = (\Phi_c)^{\mathsf{S}_\xi^t \vec{s}}(\zeta(b),\omega)$$

from above. This is a trigonometric polynomial with $\mathsf{S}_\zeta^t(b,c)$ exponential terms and all coefficients equal to one. By (3.6) and (3.23), this polynomial includes the terms

$$
\begin{aligned}
&\exp\!\big(-2\pi i\omega|p^{(b)}|_{\mathsf{S}_\xi^t \vec{s}}\big) + \exp\!\big(-2\pi i\omega|p^{(b)}v|_{\mathsf{S}_\xi^t \vec{s}}\big) \\
=\;\; &\exp\!\big(-2\pi i\omega|\xi(p^{(b)})|_{\vec{s}}\big) + \exp\!\big(-2\pi i\omega|\xi(p^{(b)})\xi(v)|_{\vec{s}}\big) \\
=\;\; &\exp\!\big(-2\pi i\omega|\xi(p^{(b)})|_{\vec{s}}\big) \cdot \big(1 + e^{-2\pi i\omega|\xi(v)|_{\vec{s}}}\big).
\end{aligned}
$$

It follows that

$$|\mathbf{M}_{\xi,\zeta}(b,c)| \leq \mathsf{S}_\zeta^t(b,c) - 2 + \big|1 + e^{-2\pi i\omega|\xi(v)|_{\vec{s}}}\big|,$$

and from the inequality $|1 + e^{2\pi i\tau}| \leq 2 - (1/2)\|\tau\|_{\mathbb{R}/\mathbb{Z}}^2, \quad \tau \in \mathbb{R}$, we obtain

$$(3.24) \qquad |\mathbf{M}_{\xi,\zeta}(b,c)| \leq \mathsf{S}_\zeta^t(b,c) - \frac{1}{2}\big\||\omega|\xi(v)|_{\vec{s}}\big\|_{\mathbb{R}/\mathbb{Z}}^2.$$

Now, for an arbitrary $\vec{x} = [x_1, \ldots, x_m]^t > \vec{0}$ and $b \in \mathcal{A}$, using (3.24) we can estimate

$$
\begin{aligned}
(\mathbf{M}_{\xi,\zeta}^{|\cdot|}\vec{x})_b &= \sum_{j=1}^m |\mathbf{M}_{\xi,\zeta}(b,j)| \cdot x_j \\
&\leq \sum_{j=1}^m \mathsf{S}_\zeta^t(b,j)x_j - \frac{1}{2}\big\|\omega|\xi(v)|_{\vec{s}}\big\|_{\mathbb{R}/\mathbb{Z}}^2 \cdot x_c \\
&\leq \left(1 - c_2\psi(\vec{x})\big\|\omega|\xi(v)|_{\vec{s}}\big\|_{\mathbb{R}/\mathbb{Z}}^2\right) \cdot \sum_{j=1}^m \mathsf{S}_\zeta^t(b,j)x_j
\end{aligned}
$$

(3.25)
$$
= \left(1 - c_2\psi(\vec{x})\big\|\omega|\xi(v)|_{\vec{s}}\big\|_{\mathbb{R}/\mathbb{Z}}^2\right) \cdot (\mathsf{S}_\zeta^t\vec{x})_b,
$$

where

$$
c_2 = \frac{1}{2m\max_{i,j}\mathsf{S}_\zeta^t(i,j)} = \frac{1}{2m\max_{i,j}Q_{ij}} \quad \text{and} \quad \psi(\vec{x}) = \frac{\min_j x_j}{\max_j x_j}.
$$

Thus,

(3.26)
$$
\mathbf{M}_{\xi,\zeta}^{|\cdot|}\vec{x} \leq \left(1 - c_2\psi(\vec{x})\big\|\omega|\xi(v)|_{\vec{s}}\big\|_{\mathbb{R}/\mathbb{Z}}^2\right) \cdot \mathsf{S}_\zeta^t\vec{x},
$$

and $v \in GR(\zeta)$ is arbitrary. We will apply the last inequality with $\xi = \zeta^{[n-1]}$ and

$$
\vec{x} = \vec{x}_n = (\mathsf{S}^{[n-1]})^t\vec{1} \in Q^t\mathbb{R}_+^m,
$$

recalling that $\mathsf{S}_{n-1} = \mathsf{S}_{\zeta_{n-1}} = Q\mathsf{S}_{\xi_{n-1}}Q$. Since the matrix Q is strictly positive, we have

$$
\psi(\vec{x}) = (\mathrm{col}(\vec{x}))^{-1} \geq (\mathrm{col}(Q^t))^{-1} \quad \text{for any } \vec{x} \in Q^t\mathbb{R}_+^m
$$

by (3.17). Note also that for any substitutions ξ_1, ξ_2 we have

$$
M_{\xi_1,\xi_2}^{|\cdot|} \leq \mathsf{S}_{\xi_2}^t
$$

by the definition (3.13). Therefore, taking (3.22) and (3.26) into account, we obtain

$$
\begin{aligned}
\mathbf{M}_n^{|\cdot|}(\mathsf{S}^{[n-1]})^t\vec{1} &\leq \mathbf{M}_{\zeta^{[n-1]}\zeta\xi_n,\zeta}^{|\cdot|} \cdot \mathbf{M}_{\zeta^{[n-1]}\zeta,\xi_n}^{|\cdot|} \cdot \mathbf{M}_{\zeta^{[n-1]},\zeta}^{|\cdot|}(\mathsf{S}^{[n-1]})^t\vec{1} \\
&\leq \mathsf{S}_\zeta^t\mathsf{S}_{\xi_n}^t\left(1 - c_1\max_{v \in GR(\zeta)}\big\|\omega|\zeta^{[n-1]}(v)|_{\vec{s}}\big\|_{\mathbb{R}/\mathbb{Z}}^2\right)\mathsf{S}_\zeta^t(\mathsf{S}^{[n-1]})^t\vec{1} \\
&= \left(1 - c_1\max_{v \in GR(\zeta)}\big\|\omega|\zeta^{[n-1]}(v)|_{\vec{s}}\big\|_{\mathbb{R}/\mathbb{Z}}^2\right)(\mathsf{S}^{[n]})^t\vec{1}.
\end{aligned}
$$

where c_1 is given by (3.19). Iterating this inequality yields

$$
(\mathbf{\Pi}_n\vec{e}_a)^{|\cdot|} \leq \prod_{n \leq N-1}\left(1 - c_1\max_{v \in GR(\zeta)}\big\|\omega|\zeta^{[n]}(v)|\big\|_{\mathbb{R}/\mathbb{Z}}^2\right) \cdot (\mathsf{S}^{[N]})^t\vec{1}.
$$

Finally, note that the maximal component of $(\mathsf{S}^{[N]})^t\vec{1}$ is the maximal column sum of $\mathsf{S}^{[N]}$, which is $\|\mathsf{S}^{[N]}\|_1$, and the proposition is proved completely. We emphasize that we used in an essential way that every $\zeta_n = \zeta\xi_n\zeta$ both starts and end with ζ. $\qquad\square$

3.5. **Cylindrical functions of higher order.** Suppose now that f on $\mathfrak{X}$ is a "cylindrical function of level ℓ", that is, its value depends only on the first ℓ edges of the path $\bar{\mathfrak{e}}$ and on the height t. It is then convenient to represent h_τ as a suspension flow with a different height function, based on the decomposition (disjoint in measure)

$$\mathfrak{X}^{\vec{s}} = \bigcup_{a \in \mathcal{A}} \mathfrak{X}_a^{(\ell)}, \quad \text{where } \mathfrak{X}_a^{(\ell)} = \{(\bar{\mathfrak{e}}, t) \in \mathfrak{X}^{\vec{s}} : \bar{\mathfrak{e}} \in X_+, \ x_\ell = F(\mathfrak{e}_{\ell+1}) = a\}.$$

The BV-transformation $\mathfrak{T}$ "changes" a vertex a at the ℓ-th level after $h_a^{(\ell)} = |\zeta^{[\ell]}(a)|$ iterates. Thus, after we enter the cylinder $\mathfrak{X}_a^{(\ell)}$, the flow h_τ stays in it for the time equal to

$$(3.27) \qquad s_a^{(\ell)} := |\zeta^{[\ell]}(a)|_{\vec{s}} = [(\mathsf{S}^{[\ell]})^t \vec{s}]_a.$$

More precisely, if $F(\mathfrak{e}_{\ell+1}) = a$ and $(\bar{\mathfrak{e}}, t) \in \mathfrak{X}^{\vec{s}}$, then

$$(3.28) \qquad \exists\, t' \in \left[0, s_a^{(\ell)}\right] \quad \text{such that} \quad (\bar{\mathfrak{e}}, t) = h_{t'}(\mathfrak{e}_1' \ldots \mathfrak{e}_\ell' \mathfrak{e}_{\ell+1}\mathfrak{e}_{\ell+2} \ldots, 0),$$

where $\mathfrak{e}_1' \ldots \mathfrak{e}_\ell'$ is the minimal path in the Bratteli diagram from the vertex a on level ℓ to the level 0. Observe that the horizontal sequence of a path $\mathfrak{e}_1' \ldots \mathfrak{e}_\ell' \mathfrak{e}_{\ell+1}\mathfrak{e}_{\ell+2} \ldots$, with $F(\mathfrak{e}_{\ell+1}) = a$, begins with $\zeta^{[\ell]}(a)$ and can be written as $\zeta^{[\ell]}(x^{(\ell)})$ for some $x^{(\ell)} \in \mathcal{A}^\mathbb{N}$. (In fact, $x^{(\ell)} = \mathfrak{h}(\sigma^\ell \mathfrak{e}) \in \mathcal{Y}^{(\ell)}$, see Section 2). To summarize this discussion, for any real-valued continuous cylindrical function f of level ℓ on X_+ there exist $c_a \in \mathbb{R}$ and $\psi_a^{(\ell)} \in C([0, s_a^{(\ell)}])$, $a \in \mathcal{A}$, such that

$$(3.29) \qquad f = \sum_{a \in \mathcal{A}} c_a f_a^{(\ell)}, \quad \text{where } f_a^{(\ell)}(\bar{\mathfrak{e}}, t) = \mathbb{1}_{\mathfrak{X}_a^{(\ell)}} \psi_a^{(\ell)}(t'), \quad \text{with } t' \text{ from } (3.28).$$

Now we can also write down a generalization of (3.7). Denote $\vec{s}^{(\ell)} = (s_a^{(\ell)})_{a \in \mathcal{A}}$ and assume $\bar{\mathfrak{e}}' = \mathfrak{e}_1' \ldots \mathfrak{e}_\ell' \mathfrak{e}_{\ell+1}\mathfrak{e}_{\ell+2} \ldots$, with $F(\mathfrak{e}_{\ell+1}) = a$. Then

$$(3.30) \qquad S_R^{(\bar{\mathfrak{e}}', 0)}(f_a^{(\ell)}, \omega) = \widehat{\psi}_a^{(\ell)}(\omega) \cdot \Phi_a^{\vec{s}^{(\ell)}}\big(x^{(\ell)}[0, N-1], \omega\big) \quad \text{for } R = \big|x^{(\ell)}[0, N-1]\big|_{\vec{s}^{(\ell)}},$$

where $(x_n)_{n \geq 0} = \zeta^{[\ell]}(x_n^{(\ell)})_{n \geq 0}$ and $(x_n)_{n \in \mathbb{Z}} = \mathfrak{h}(\bar{\mathfrak{e}})$. In the next corollary we extend Proposition 3.4 to cylindrical functions of level ℓ.

Corollary 3.5. *Under the assumptions of Proposition 3.4, for any $\ell \geq 1$, $a, b \in \mathcal{A}$, $n \geq \ell + 1$, $\vec{s} > \vec{0}$, and $\omega \in \mathbb{R}$, we have*

$$(3.31) \qquad \big|\Phi_a^{\vec{s}^{(\ell)}}(\zeta^{[\ell+1,n]}(b), \omega)\big| \leq \|\mathsf{S}^{[\ell+1,n]}\|_1 \cdot \prod_{\ell+1 \leq k \leq n-1}\left(1 - c_1 \cdot \max_{v \in GR(\zeta)} \|\omega|\zeta^{[k]}(v)|_{\vec{s}}\|_{\mathbb{R}/\mathbb{Z}}^2\right),$$

where c_1 is given by (3.19).

Proof. It is immediate from Proposition 3.4 by shifting the indices that

$$\big|\Phi_a^{\vec{s}^{(\ell)}}(\zeta^{[\ell+1,n]}(b), \omega)\big| \ \leq \ \|\mathsf{S}^{[\ell+1,n]}\|_1 \cdot \prod_{\ell+1 \leq k \leq n-1}\left(1 - c_1 \cdot \max_{v \in GR(\zeta)} \|\omega|\zeta^{[\ell+1,k]}(v)|_{\vec{s}^{(\ell)}}\|_{\mathbb{R}/\mathbb{Z}}^2\right) \cdot$$

It remains to note that

$$
\begin{aligned}
|\zeta^{[\ell+1,k]}(v)|_{\vec{s}^{(\ell)}} &= \langle \vec{\ell}(\zeta^{[\ell+1,k]}(v)), \vec{s}^{(\ell)} \rangle \\
&= \langle \mathsf{S}^{[\ell+1,k]}\vec{\ell}(v), (\mathsf{S}^\ell)^t \vec{s} \rangle \\
&= \langle \mathsf{S}^\ell \mathsf{S}^{[\ell+1,k]}\vec{\ell}(v), \vec{s} \rangle \\
&= \langle \mathsf{S}^{[k]}\vec{\ell}(v), \vec{s} \rangle \\
&= \langle \vec{\ell}(\zeta^{[k]}(v)), \vec{s} \rangle = |\zeta^{[k]}(v)|_{\vec{s}}.
\end{aligned}
$$

$\square$

Next we need to pass from the exponential sum corresponding to the word $\zeta^{[\ell+1,n]}(b)$ to the one corresponding to a general word in the space $\mathcal{Y}^{(\ell)}$. To this end, we will use the well-known prefix-suffix decomposition.

Lemma 3.6. *Let $x^{(\ell)} \in \mathcal{Y}^{(\ell)}$ and $N \geq 1$. Then*

$$(3.32) \quad x^{(\ell)}[0, N-1] = \zeta^{[\ell+1]}(u_{\ell+1})\zeta^{[\ell+1,\ell+2]}(u_{\ell+2})\ldots\zeta^{[\ell+1,n]}(u_{\ell+n})\zeta^{[\ell+1,n]}(v_{\ell+n})\ldots\zeta^{[\ell+1]}(v_{\ell+1}),$$

where u_j, v_j, $j = \ell+1, \ldots, \ell+n$, are respectively proper suffixes and prefixes of the words $\zeta_{j+1}(b)$, $b \in \mathcal{A}$. The words u_j, v_j may be empty, except that at least one of $u_{\ell+n}, v_{\ell+n}$ is nonempty. Moreover,

$$(3.33) \qquad\qquad \min_{b \in \mathcal{A}} |\zeta^{[\ell+1,n]}(b)| \leq N \leq 2\max_{b \in \mathcal{A}} |\zeta^{[\ell+1,n+1]}(b)|.$$

Proof. This is immediate from the description of $\mathcal{Y}^{(\ell)}$ at the end of Section 2. $\square$

Proposition 3.7. *Under the assumptions of Proposition 3.4, for any $\ell \geq 1$, $a \in \mathcal{A}$, $N \in \mathbb{N}$, $\vec{s} > \vec{0}$, $x^{(\ell)} \in \mathcal{Y}^{(\ell)}$, and $\omega \in \mathbb{R}$, we have,*

$$(3.34) \quad \left|\Phi_a^{\vec{s}^{(\ell)}}\big(x^{(\ell)}[0, N-1], \omega\big)\right| \leq 2\sum_{j=\ell}^{n} \|\mathsf{S}^{[\ell+1,j]}\|_1 \cdot \|\mathsf{S}_{j+1}\|_1 \cdot \prod_{\ell+1 \leq k \leq j-1} \Big(1 - c_1 \cdot \max_{v \in GR(\zeta)} \|\omega|\zeta^{[k]}(v)|_{\vec{s}}\|_{\mathbb{R}/\mathbb{Z}}^2\Big),$$

where c_1 is given by (3.19) and $n \in \mathbb{N}$ is such that (3.33) holds. Here we let $\mathsf{S}^{[\ell+1,\ell]} =: I$.

Proof. We use Lemma 3.6, and apply Corollary 3.5 to each term. The factor $2\|\mathsf{S}_{j+1}\|_1$ in (3.34) appears, because $|u_j|, |v_j| \leq \max_b |\zeta_j(b)| = \|\mathsf{S}_{j+1}\|_1$ in (3.32). $\square$

4. RANDOM BV-TRANSFORMATIONS: STATEMENT OF THE THEOREM AND PLAN OF THE PROOF

Here we consider dynamical systems generated by a *random* sequence of Markov compacta. In order to state our results, we need some preparation; specifically, the Oseledets Theorem.

Recall that $\mathfrak{G}$ denotes the set of all oriented graphs on m vertices such that there is an edge starting at every vertex and an edge ending at every vertex (we allow loops and multiple edges).

We also assume that each graph is equipped with a Vershik ordering. Let Ω be the space of sequences of graphs:

$$\Omega = \{\mathbf{a} = \ldots \mathbf{a}_{-n} \ldots \mathbf{a}_0.\mathbf{a}_1 \ldots \mathbf{a}_n \ldots, \ \ \mathbf{a}_i \in \mathfrak{G}, \ i \in \mathbb{Z}\}.$$

For $\mathbf{a} \in \Omega$ we denote by $X(\mathbf{a})$ the Markov compactum corresponding to $\mathbf{a}$ according to the rule $\Gamma_n = \mathbf{a}_n, \ \ n \in \mathbb{Z}$, and let σ the left shift on Ω. We also consider the corresponding one-sided compactum $X_+(\mathbf{a})$. For a word $\mathbf{q} = \mathbf{q}_1 \ldots \mathbf{q}_k \in \mathfrak{G}^k$ we can "concatenate" the graphs to obtain the "aggregated" graph $\Gamma_{\mathbf{q}}$, also belonging to $\mathfrak{G}$. By the definition of incidence matrix, we have

$$A(\mathbf{q}) := A(\Gamma_{\mathbf{q}}) = A(\mathbf{q}_k) \cdot \ldots \cdot A(\mathbf{q}_1).$$

Since the graphs are equipped with the Vershik ordering, we also have a corresponding sequence of substitutions, so that $\zeta(\mathbf{q}) = \zeta(\mathbf{q}_1) \ldots \zeta(\mathbf{q}_k)$. We will also need a "2-sided cylinder set":

$$[\mathbf{q}.\mathbf{q}] = \{\mathbf{a} \in \Omega : \ \mathbf{a}_{-k+1} \ldots \mathbf{a}_0 = \mathbf{a}_1 \ldots \mathbf{a}_k = \mathbf{q}\}.$$

Following [11], we say that the word $\mathbf{q} = q_1 \ldots q_k$ is "simple" if for all $2 \leq i \leq k$ we have $q_i \ldots q_k \neq q_1 \ldots q_{k-i+1}$. If the word $\mathbf{q}$ is simple, two occurrences of $\mathbf{q}$ in the sequence $\mathbf{a}$ cannot overlap. Let $\mathbb{P}$ be an ergodic σ-invariant probability measure on Ω satisfying the following

Conditions:

   (C1) *There exists a word* $\mathbf{q} \in \mathfrak{G}^k$ *such that all the entries of the matrix* $A(\mathbf{q})$ *are positive and*

(4.1) $$\mathbb{P}\left([\mathbf{q}.\mathbf{q}]\right) > 0.$$

   (C2) *The matrices* $A(\mathbf{a}_n)$ *are almost surely invertible with respect to* $\mathbb{P}$.
   (C3) *The functions* $\mathbf{a} \mapsto \log(1 + \|A^{\pm 1}(\mathbf{a}_1)\|)$ *are integrable.*
(Here and below $\|A\|$ denotes the Euclidean operator norm of the matrix.)

   Observe that **(C3)**, together with the Birkhoff ergodic theorem, immediately gives

(4.2) $$\lim_{n \to \infty} n^{-1} \log(1 + \|A(\mathbf{a}_n)\|) = 0 \ \text{ for } \mathbb{P}\text{-a.e. } \mathbf{a} \in \Omega.$$

We obtain a measurable cocycle $\mathbb{A} : \Omega \to GL(m, \mathbb{R})$, defined by $\mathbb{A}(\mathbf{a}) = A(\mathbf{a}_1)$, called the *renormalization cocycle*. Denote

(4.3) $$\mathbb{A}(n, \mathbf{a}) = \begin{cases} \mathbb{A}(\sigma^{n-1}\mathbf{a}) \cdot \ldots \cdot \mathbb{A}(\mathbf{a}), & n > 0; \\ Id, & n = 0; \\ \mathbb{A}^{-1}(\sigma^{-n}\mathbf{a}) \cdot \ldots \cdot \mathbb{A}^{-1}(\sigma^{-1}\mathbf{a}), & n < 0, \end{cases}$$

so that

$$\mathbb{A}(n, \mathbf{a}) = A(\mathbf{a}_n) \cdots A(\mathbf{a}_1), \ n \geq 1.$$

As in Section 2, we consider the sequence of substitutions $\zeta(\mathbf{a}_k)$, $\mathbf{a} \in \Omega$, $k \in \mathbb{Z}$, and their substitution matrices $\mathsf{S}_{\zeta_k(\mathbf{a})} = A^t(\mathbf{a}_k)$. (Recall that all graphs $\mathbf{a}_k$ are equipped with a Vershik ordering.) Thus

$$\mathbb{A}(n, \mathbf{a}) = \mathsf{S}^t(\mathbf{a}_n \ldots \mathbf{a}_1) = \mathsf{S}^t_{\zeta(\mathbf{a}_1 \ldots \mathbf{a}_n)} \ n \geq 1.$$

By the Oseledets Theorem [26] (for a detailed survey, see Barreira-Pesin [6]), there exist Lyapunov exponents $\theta_1 > \theta_2 > \ldots > \theta_r$ and, for $\mathbb{P}$-a.e. $\mathbf{a} \in \Omega$, a direct-sum decomposition

(4.4)
$$\mathbb{R}^m = E_{\mathbf{a}}^1 \oplus \cdots \oplus E_{\mathbf{a}}^r$$

that depends measurably on $\mathbf{a} \in \Omega$ and satisfies the following:

(i) for $\mathbb{P}$-a.e. $\mathbf{a} \in \Omega$, any $n \in \mathbb{Z}$, and any $i = 1, \ldots, r$ we have

$$\mathbb{A}(n, \mathbf{a}) E_{\mathbf{a}}^i = E_{\sigma^n \mathbf{a}}^i;$$

(ii) for any $v \in E_{\mathbf{a}}^i$, $v \neq 0$, we have

$$\lim_{|n| \to \infty} \frac{\log \|\mathbb{A}(n, \mathbf{a}) v\|}{n} = \theta_i.$$

(iii) $\lim_{|n| \to \infty} \frac{1}{n} \log \angle \left(\bigoplus_{i \in I} E_{\sigma^n \mathbf{a}}^i, \bigoplus_{j \in J} E_{\sigma^n \mathbf{a}}^j \right) = 0$ whenever $I \cap J = \emptyset$.

Let $P_{\mathbf{a}}^i$ be the projection to $E_{\mathbf{a}}^i$ arising from (4.4). Denote by σ_f the spectral measure for the system $(\mathfrak{X}^{\vec{s}}, h_t)$ with the test function f (assuming the system is uniquely ergodic). Now we can state our theorem.

Theorem 4.1. *Let $(\Omega, \mathbb{P}, \sigma)$ be an invertible ergodic measure-preserving system satisfying conditions* **(C1)-(C3)** *above. Consider the cocycle $\mathbb{A}(n, \mathbf{a})$ defined by (4.3). Assume that*

(a) *the Lyapunov spectrum satisfies*

$$\theta_1 > \theta_2 > 0 > \theta_3 > \ldots,$$

and the two top exponents are simple (i.e. $\dim(E_{\mathbf{a}}^1) = \dim(E_{\mathbf{a}}^2) = 1$ for $\mathbb{P}$-a.e. $\mathbf{a}$);

(b) *there exists a simple word $\mathbf{q} \in \mathfrak{G}^k$ for some $k \in \mathbb{N}$, such that all the entries of the matrix $A(\mathbf{q})$ are strictly positive and $\mathbb{P}([\mathbf{q}.\mathbf{q}]) > 0$;*

(c) *there exist "good return words" $\{u_j\}_{j=1}^m$ for $\zeta = \zeta(\mathbf{q})$ (see Definition 3.3), such that $\{\vec{\ell}(u_j)\}_{j=1}^m$ is a basis for $\mathbb{R}^m$;*

(d) *Let $\ell_{\mathbf{q}}(\mathbf{a})$ be the "negative" waiting time until the first appearance of $\mathbf{q}.\mathbf{q}$, i.e.*

$$\ell_{\mathbf{q}}(\mathbf{a}) = \min\{n \geq 1 : \sigma^{-n} \mathbf{a} \in [\mathbf{q}.\mathbf{q}]\}.$$

Let $\mathbb{P}(\mathbf{a}|\mathbf{a}^+)$ be the conditional distribution on the set of $\mathbf{a}$'s conditioned on the future $\mathbf{a}^+ = \mathbf{a}_1 \mathbf{a}_2 \ldots$ We assume that there exist $\varepsilon > 0$ and $1 < C < \infty$ such that

(4.5)
$$\int_{[\mathbf{q}.\mathbf{q}]} \left\| \mathbb{A}(\ell_{\mathbf{q}}(\mathbf{a}), \sigma^{-\ell_{\mathbf{q}}(\mathbf{a})} \mathbf{a}) \right\|^{\varepsilon} \, d\mathbb{P}(\mathbf{a}|\mathbf{a}^+) \leq C \quad \text{for all } \mathbf{a}^+ \text{ starting with } \mathbf{q}.$$

Then there exists $\gamma > 0$ such that for $\mathbb{P}$-a.e. $\mathbf{a} \in \Omega$ the following holds:

Let $(X_+, \mathfrak{T})$ be the Bratteli-Vershik system corresponding to $\mathbf{a}^+$, which is uniquely ergodic. Let $(\mathfrak{X}^{\vec{s}}, h_t)$ be the suspension flow over $(X_+, \mathfrak{T})$ under the piecewise-constant roof function determined by $\vec{s}$. Then for all $\beta > 0$ and $B > 1$ there exists $r_0 = r_0(\mathbf{a}, \beta, B) > 0$, such that for Lebesgue-a.e. $\vec{s}$, with $\|\vec{s}\|_1 = 1$ and $\min_{j=1,2} |P_{\mathbf{a}}^j(\vec{s})| \geq \beta$, for any $f \in \mathrm{Lip}_w^+(\mathfrak{X}^{\vec{s}})$,

$$(4.6) \qquad \sigma_f(B(\omega, r)) \leq C(\mathbf{a}, \|f\|_L) \cdot r^\gamma \quad \text{for all} \ \ \omega \in [B^{-1}, B] \ \ \text{and} \ \ 0 < r < r_0,$$

with the constant depending only on $\mathbf{a}$ and $\|f\|_L$.

Remarks. 1. It is clear that condition **(C1)** follows from assumption (b), but we chose to state **(C1)** explicitly, since this is the condition which appears in the literature and implies unique ergodicity. The unique ergodicity of the system $(X_+, \mathfrak{T})$ for $\mathbb{P}$-a.e. $\mathbf{a}$ under the given assumptions is well-known and goes back to the work of Furstenberg [20] (see the beginning of Section 2).

2. The assumption that $\mathbf{q}$ is a simple word ensures that occurrences of $\mathbf{q}$ do not overlap. Then we have

$$(4.7) \qquad \mathbb{A}(\ell_{\mathbf{q}}(\mathbf{a}), \sigma^{-\ell_{\mathbf{q}}(\mathbf{a})}\mathbf{a}) = A(\mathbf{q})A(\mathbf{p})A(\mathbf{q}),$$

for some $\mathbf{p} \in \mathfrak{G}$ (possibly trivial). For our application, it will be easy to make sure that $\mathbf{q}$ is simple, as we show in Section 11, unlike in the paper [11], where additional efforts were needed to achieve the desired aims.

The scheme of the proof is as follows: first we reduce the theorem to the case where all the symbols have the form $\mathbf{a}_n = \mathbf{q}\mathbf{p}_n\mathbf{q}$. This is done by considering the first return map to the cylinder set $[\mathbf{q}.\mathbf{q}]$. Then we apply Proposition 3.7, with the goal to use Lemma 3.1. In order to achieve the desired estimate, roughly speaking, we need to show that for $\mathbb{P}$-a.e. sequence of substitutions, for Lebesgue a.e. $\vec{s}$, the distance from $\omega|\zeta^{[n]}(v_n)|_{\vec{s}}$ to the nearest integer (for some choice of a good return word v_n which depends on n) is bounded away from zero for a positive frequency of n's (uniformly in ω bounded away from zero and infinity). The proof splits into two parts, separating the two "almost every". The first part is probabilistic, showing that certain assumptions on the sequence of substitutions $\zeta(\mathbf{p}_n)$ hold $\mathbb{P}$-almost surely. In the second part we fix a typical sequence $\zeta(\mathbf{p}_n)$ and obtain estimates for a.e. $\vec{s}$. This is done using the "Erdős-Kahane argument."

5. REDUCTION

In this short section, we show that Theorem 4.1 reduces to the case when

$$(5.1) \qquad \mathbf{a}_n = \mathbf{q}\mathbf{p}_n\mathbf{q} \quad \text{for all} \ \ n \in \mathbb{Z},$$

where $\mathbf{q}$ is a fixed graph with fixed Vershik ordering, such that its incidence matrix is strictly positive, and $\mathbf{p}_n$ is arbitrary. In the next theorem we use the same notation as in Theorem 4.1.

Theorem 5.1. *Let $(\Omega_{\mathbf{q}}, \mathbb{P}, \sigma)$ be an invertible ergodic measure-preserving system of the form (5.1) satisfying conditions* **(C1)-(C3)** *from Section 4. Consider the cocycle $\mathbb{A}(n, \mathbf{a})$ defined by (4.3). Assume that*

(a′) *the Lyapunov spectrum satisfies*

$$\theta_1 > \theta_2 > 0 > \theta_3 > \dots,$$

and the two top exponents are simple;

(b′) *the substitution $\zeta = \zeta(\mathbf{q})$ is such that its substitution matrix $\mathsf{S}_\zeta = Q$ has strictly positive entries;*

(c′) *there exist "good return words" $\{u_j\}_{j=1}^m$ for $\zeta = \zeta(\mathbf{q})$ (see Definition 3.3), such that $\{\vec{\ell}(u_j)\}_{j=1}^m$ is a basis for $\mathbb{R}^m$;*

(d′) *there exist $\varepsilon > 0$ and $1 < C < \infty$ such that*

$$(5.2) \qquad \int_{\Omega_{\mathbf{q}}} \|A(\mathbf{a}_0)\|^\varepsilon \, d\mathbb{P}(\mathbf{a}|\mathbf{a}^+) \leq C \quad \text{for all } \mathbf{a}^+.$$

Then there exists $\gamma > 0$ such that for $\mathbb{P}$-a.e. $\mathbf{a} \in \Omega_{\mathbf{q}}$ the following holds:

Let $(X_+, \mathfrak{T})$ be the Bratteli-Vershik system corresponding to $\mathbf{a}^+$, which is uniquely ergodic. Let $(\mathfrak{X}^{\vec{s}}, h_t)$ be the suspension flow over $(X_+, \mathfrak{T})$ under the piecewise-constant roof function determined by $\vec{s}$. Then for all $\beta > 0$ and $B > 1$ there exists $r_0 = r_0(\mathbf{a}, \beta, B) > 0$, such that for Lebesgue-a.e. $\vec{s}$, with $\|\vec{s}\|_1 = 1$ and $\min_{j=1,2} |P_{\mathbf{a}}^j(\vec{s})| \geq \beta$, for any $f \in \mathrm{Lip}_w^+(\mathfrak{X}^{\vec{s}})$,

$$(5.3) \qquad \sigma_f(B(\omega, r)) \leq C(\mathbf{a}, \|f\|_L) \cdot r^\gamma \quad \text{for all } \omega \in [B^{-1}, B] \text{ and } 0 < r < r_0,$$

with the constant depending only on $\mathbf{a}$ and $\|f\|_L$.

Remark. If we assume that $\mathbb{P}$ is "quasi-Bernoulli", i.e. it satisfies the "bounded distortion property" of [5], then (5.2) can be replaced by the "unconditional" estimate $\int_{\Omega_{\mathbf{q}}} \|A(\mathbf{a}_0)\|^\varepsilon \, d\mathbb{P}(\mathbf{a}) \leq C$. However, we prefer the current formulation.

Proof of Theorem 4.1 assuming Theorem 5.1. Given an ergodic system $(\Omega, \mathbb{P}, \sigma)$ from the statement of Theorem 4.1, we consider the induced system on the cylinder set $\Omega_{\mathbf{q}} := [\mathbf{q}.\mathbf{q}]$. Then symbolically we can represent elements of $\Omega_{\mathbf{q}}$ as sequences satisfying (5.1). Denote by $\mathbb{P}_{\mathbf{q}}$ the induced (conditional) measure on $\Omega_{\mathbf{q}}$. Since $\mathbb{P}([\mathbf{q}.\mathbf{q}]) > 0$, standard results in Ergodic Theory imply that the resulting induced system $(\Omega_{\mathbf{q}}, \mathbb{P}_{\mathbf{q}}, \sigma)$ is also ergodic and the associated cocycle has the same properties of the Lyapunov spectrum (with the values of the Lyapunov exponents multiplied by $1/\mathbb{P}([\mathbf{q}.\mathbf{q}])$); that is, (a′) holds. The properties (b′) and (c′) follow from (b) and (c) automatically. Finally, note that (5.2) is identical to (4.5). On the level of Bratteli-Vershik diagrams this corresponds to the "aggregation-telescoping procedure", which results in a naturally isomorphic Bratteli-Vershik system. Observe also that a weakly-Lipschitz function on Ω induces a weakly-Lipschitz function on $\Omega_{\mathbf{q}}$ without increase of the norm $\|f\|_L$, see Section 2.1. Thus, Theorem 5.1 applies, and the reduction is complete. $\qquad \square$

The next five sections are devoted to the proof of Theorem 5.1. In the last Section 11 we return to the setting of Theorem 4.1 and deduce Theorem 1.1 from it.

6. Exponential tails

For $\mathbf{a} \in \Omega_{\mathbf{q}}$ we consider the sequence of substitutions $\zeta(\mathbf{a}_n)$, $n \geq 1$. In view of (5.1), we have

$$\zeta(\mathbf{a}_n) = \zeta(\mathbf{q})\zeta(\mathbf{p}_n)\zeta(\mathbf{q}).$$

Recall that

$$A(\mathbf{p}_n) = \mathsf{S}^t_{\zeta(\mathbf{p}_n)}.$$

Denote

(6.1) $$W_n = W_n(\mathbf{a}) := \log \|A(\mathbf{a}_n)\| = \log \|Q^t A(\mathbf{p}_n) Q^t\|.$$

Since all the matrices in the product have non-negative integer entries and a.s. invertible, and the first and last one are equal to Q with strictly positive entries, we have $W_n > 0$, $n \geq 1$, for $\mathbb{P}$-a.e. $\mathbf{a}$. This will always be assumed below, without loss of generality.

Proposition 6.1. *Under the assumptions of Theorem 5.1, there exists a positive constant L_1 such that for $\mathbb{P}$-a.e. $\mathbf{a}$, the following holds: for any $\delta > 0$, for all N sufficiently large $(N \geq N_0(\mathbf{a}, \delta))$,*

(6.2) $$\max \left\{ \sum_{n \in \Psi} W_n : \ \Psi \subset \ \{1, \ldots, N\}, \ |\Psi| \leq \delta N \right\} \leq L_1 \cdot \log(1/\delta) \cdot \delta N.$$

We will prove the proposition at the end of the section, but first point out the following.

Remark 6.2. *It follows from (6.2) that for any $\widetilde{\delta} > 0$, for all n sufficiently large,*

(6.3) $$W_n \leq \widetilde{\delta} n.$$

Indeed, in (6.2) we just need to take $\delta > 0$ such that $L_1 \cdot \log(1/\delta) \cdot \delta < \widetilde{\delta}$, and then $\Psi = \{n\}$, which clearly satisfies the condition $1 = |\Psi| \leq \delta N$ for N sufficiently large.

As the referee pointed out, this also follows directly from the Birkhoff Ergodic Theorem, since $\frac{1}{n}(W_1 + \cdots + W_n) \to \int_{\Omega_{\mathbf{q}}} \log \|A(\mathbf{a})\| \, d\mathbb{P} < \infty$ for a.e. $\mathbf{a} \in \Omega_{\mathbf{q}}$.

Lemma 6.3. *We have for all N and n, and for any (deterministic!) increasing sequence $1 \leq j_1 < j_2 < \ldots < j_n$:*

(6.4) $$\mathbb{P}\left[\sum_{i=1}^{n} W_{j_i} \geq Kn\right] \leq \exp(-\varepsilon K n/2) \ \ for \ \ K \geq \frac{2 \log C}{\varepsilon},$$

where $\varepsilon > 0$ and $C > 1$ are the constants from (4.5).

This is a standard large deviation result, but we provide a proof for completeness. Thanks to Chris Hoffman who showed us the argument.

Proof. Let $X_i = W_{j_i} - K$. Then $\mathbb{P}\left[\sum_{i=1}^{n} W_{j_i} \geq Kn\right] = \mathbb{P}\left[\sum_{i=1}^{n} X_i \geq 0\right]$. Observe that the values of $W_{j_2}, \ldots W_{j_n}$ are determined by the "future" of $\sigma\mathbf{a}$, that is, by $(\sigma\mathbf{a})^+ = \mathbf{a}_2\mathbf{a}_3\ldots$, hence by (4.5) we have

$$\mathbb{E}\left[e^{\varepsilon W_{j_1}} \mid W_{j_2}, \ldots, W_{j_n}\right] < C.$$

Therefore,

(6.5) $$\mathbb{E}\left[e^{\varepsilon X_1} \mid X_2, \ldots, X_n\right] < Ce^{-\varepsilon K} \leq e^{-\varepsilon K/2},$$

provided $K \geq 2\varepsilon^{-1}\log C$. Let $S_\ell = \sum_{i=n-\ell+1}^{n} X_i$. Now,

$$
\begin{aligned}
\mathbb{E}\left[e^{\varepsilon S_n}\right] &= \sum_b \mathbb{E}\left[e^{\varepsilon S_n} \mid e^{\varepsilon S_{n-1}} = b\right] \cdot \mathbb{P}\left[e^{\varepsilon S_{n-1}} = b\right] \\
&= \sum_b b \cdot \mathbb{E}\left[e^{\varepsilon X_n} \mid e^{\varepsilon S_{n-1}} = b\right] \cdot \mathbb{P}\left[e^{\varepsilon S_{n-1}} = b\right] \\
&\leq e^{-\varepsilon K/2}\sum_b b \cdot \mathbb{P}\left[e^{\varepsilon S_{n-1}} = b\right] = e^{-\varepsilon K/2}\mathbb{E}\left[e^{\varepsilon S_{n-1}}\right],
\end{aligned}
$$

taking (6.5) into account. Iterating the last inequality yields

$$\mathbb{E}\left[e^{\varepsilon S_n}\right] \leq e^{-\varepsilon Kn/2},$$

and since $\mathbb{P}\left[S_n \geq 0\right] \leq \mathbb{E}\left[e^{\varepsilon S_n}\right]$, the estimate (6.4) is proved. $\square$

Proof of Proposition 6.1. Consider the event

$$\mathcal{W}(N, \delta, K) = \left\{ \max_{\substack{\Psi \subset \{1,\ldots,N\} \\ |\Psi| \leq \delta N}} \sum_{n \in \Psi} W_n \geq K(\delta N) \right\}$$

Then we have for $K \geq 2\log C/\varepsilon$,

$$
\begin{aligned}
\mathbb{P}\left(\mathcal{W}(N, \delta, K)\right) &\leq \sum_{\substack{\Psi \subset \{1,\ldots,N\} \\ |\Psi| \leq \delta N}} \mathbb{P}\left[\sum_{n \in \Psi} W_n \geq K(\delta N)\right] \\
&\leq \sum_{i \leq \delta N} \binom{N}{i} e^{-\varepsilon K(\delta N)/2},
\end{aligned}
$$

in view of Lemma 6.3. By Stirling, there exists $C' > 1$ such that

(6.6) $$\sum_{i \leq \delta N} \binom{N}{i} \leq \exp\left[C'\delta\log(1/\delta)N\right] \quad \text{for} \quad \delta < e^{-1} \quad \text{and all} \quad N > 1.$$

Therefore,

$$\sum_{i \leq \delta N} \binom{N}{i} \leq \exp[-\varepsilon K(\delta N)/4] \quad \text{for} \quad K = \frac{4C'}{\varepsilon}\log(1/\delta),$$

whence, by Borel-Cantelli, the event $\mathcal{W}(N, \delta, L_1 \log(1/\delta))$ does not occur for all N sufficiently large, with

$$L_1 = \varepsilon^{-1} \max(4C', 2 \log C),$$

which means that condition (6.2) holds. $\hfill\square$

7. Estimating twisted Birkhoff integrals

In this section we continue to work with a $\mathbb{P}$-generic 2-sided sequence $\mathbf{a} \in \Omega_\mathbf{q}$. Under the assumptions of Theorem 5.1, for $\mathbb{P}$-a.e. $\mathbf{a}$, the sequence of substitutions $\zeta(\mathbf{a}_n)$, $n \in \mathbb{Z}$, satisfies several conditions. First of all, we can assume that the point $\mathbf{a}$ is generic for the Oseledets Theorem; that is, assertions (i)-(iii) from Section 4 hold. We further assume that the conclusions of Proposition 6.1 hold. Recall that $\zeta(\mathbf{a}_n) = \zeta(\mathbf{q})\xi_n\zeta(\mathbf{q})$, where $Q = \mathsf{S}_\zeta$ is a strictly positive matrix. Below we denote by $O_Q(1)$ a generic constant which depends only on $Q = \mathsf{S}(\zeta)$ and which may be different from line to line.

Proposition 7.1. *Suppose that the conditions of Theorem 5.1 are satisfied. Then for $\mathbb{P}$-a.e. $\mathbf{a} \in \Omega$, for any $\eta \in (0,1)$, there exists $\ell_\eta = \ell_\eta(\mathbf{a}) \in \mathbb{N}$, such that for all $\ell \geq \ell_\eta$ and any bounded cylindrical function $f^{(\ell)}$ of level ℓ, for any $(\overline{\mathfrak{c}}, t) \in \mathfrak{X}^{\vec{s}}$, with $\overline{\mathfrak{c}} \in X_+(\mathbf{a})$, and $\omega \in \mathbb{R}$,*

$$(7.1) \quad |S_R^{(\overline{\mathfrak{c}},t)}(f^{(\ell)}, \omega)| \leq O_Q(1) \cdot \|f^{(\ell)}\|_\infty \left(R^{1/2} + R^{1+\eta} \prod_{\ell+1 \leq k < \frac{\log R}{4\theta_1}} \left(1 - c_1 \cdot \max_{v \in GR(\zeta)} \left\| |\omega| \zeta^{[k]}(v)|_{\vec{s}} \right\|_{\mathbb{R}/\mathbb{Z}}^2 \right) \right),$$

for all $R \geq e^{8\theta_1 \ell}$.

Remark 7.2. By (3.5) we have

$$\| |\omega| \zeta^{[n]}(v)|_{\vec{s}} \|_{\mathbb{R}/\mathbb{Z}} = \| \langle \vec{\ell}(v), \omega(\mathsf{S}^{[n]})^t \vec{s} \rangle \|_{\mathbb{R}/\mathbb{Z}} = \| \langle \vec{\ell}(v), \mathbb{A}(n, \mathbf{a})(\omega\vec{s}) \rangle \|_{\mathbb{R}/\mathbb{Z}}.$$

In fact, our assumption (namely, condition (c') in Theorem 5.1) is that the substitution ζ possesses m good return words $v_1, \ldots, v_m$ such that their population vectors $\vec{\ell}(v_1), \ldots \vec{\ell}(v_m)$ form a basis of $\mathbb{R}^m$. Observe that $\langle \vec{\ell}(v_j), \vec{x} \rangle$, for $j = 1, \ldots, m$, are the coordinates of a vector $\vec{x} \in \mathbb{R}^m$ with respect to the basis dual to $\{\vec{\ell}(v_1), \ldots \vec{\ell}(v_m)\}$. Let Γ be the free Abelian group generated by $\vec{\ell}(v_1), \ldots \vec{\ell}(v_m)$. Then $\Gamma < \mathbb{Z}^m$ is a full rank lattice. Let $\widehat{\Gamma}$ be the dual lattice. Observe that

$$C_\zeta^{-1} \|\vec{x}\|_{\mathbb{R}^m/\widehat{\Gamma}} \leq \max_{j \leq m} \| \langle \vec{\ell}(v_j), \vec{x} \rangle \|_{\mathbb{R}/\mathbb{Z}} \leq C_\zeta \|\vec{x}\|_{\mathbb{R}^m/\widehat{\Gamma}},$$

with $C_\zeta > 1$ depending only on $\vec{\ell}(v_1), \ldots \vec{\ell}(v_m)$. Thus, estimating the product in (7.1) is equivalent to estimating

$$(7.2) \qquad \prod_{\ell+1 \leq k < \frac{\log R}{4\theta_1}} \left(1 - \widetilde{c}_1 \cdot \left\| \mathbb{A}(k, \mathbf{a})(\omega\vec{s}) \right\|_{\mathbb{R}^m/\widehat{\Gamma}}^2 \right),$$

making it similar to the expression appearing in the Veech criterion alluded to in Section 1.3. However, although the form of (7.2) may be more appealing, for technical reasons we prefer to work with the expression in (7.1).

Proof of Proposition 7.1. Without loss of generality we can assume that $f^{(\ell)} = f_a^{(\ell)}$ for some $a \in \mathcal{A}$, as in (3.29) and find $\overline{\mathfrak{e}}'$ and t' as in (3.28). Since $(\overline{\mathfrak{e}}, t) = h_{t'}(\overline{\mathfrak{e}}', 0)$, we have

$$
\begin{aligned}
|S_R^{(\overline{\mathfrak{e}},t)}(f_a^{(\ell)}, \omega)| &= \left| \int_0^R e^{-2\pi i \omega \tau} f_a^{(\ell)} \circ h_{\tau+t'}(\overline{\mathfrak{e}}', 0)\, d\tau \right| \\
&= \left| \int_{t'}^{R+t'} e^{-2\pi i \omega \tau} f_a^{(\ell)} \circ h_\tau(\overline{\mathfrak{e}}', 0)\, d\tau \right|.
\end{aligned}
$$

Recall that $\vec{s}^{(\ell)} = (\mathsf{S}^{[\ell]})^t \vec{s}$, and we let $s_{\max}^{(\ell)}$ and $s_{\max}^{(\ell)}$ be the maximal and minimal components of the vector $\vec{s}^{(\ell)}$, respectively. Note that $|t'| \leq s_{\max}^{(\ell)}$, so we obtain

$$(7.3) \qquad \left| S_R^{(\overline{\mathfrak{e}},t)}(f_a^{(\ell)}, \omega) - S_R^{(\overline{\mathfrak{e}}',0)}(f_a^{(\ell)}, \omega) \right| \leq 2\|f^{(\ell)}\|_\infty s_{\max}^{(\ell)}.$$

Next, consider $x^{(\ell)} \in \mathcal{Y}^{(\ell)}$ as in (3.30) and take the maximal N such that $R' := |x^{(\ell)}[0, N-1]|_{\vec{s}^{(\ell)}} \leq R$. Then $|R - R'| \leq s_{\max}^{(\ell)}$, hence

$$(7.4) \qquad \left| S_R^{(\overline{\mathfrak{e}}',0)}(f_a^{(\ell)}, \omega) - S_{R'}^{(\overline{\mathfrak{e}}',0)}(f_a^{(\ell)}, \omega) \right| \leq \|f^{(\ell)}\|_\infty s_{\max}^{(\ell)},$$

and for $S_{R'}^{(\overline{\mathfrak{e}}',0)}(f_a^{(\ell)}, \omega)$ the formula in (3.30) applies (with R replaced by R'). Thus, the combined error in the above estimates (7.3), (7.4) is bounded by $3\|f^{(\ell)}\|_\infty \cdot s_{\max}^{(\ell)}$. By Oseledets Theorem, we can make sure that ℓ_η is such that

$$(7.5) \qquad \left| \ell^{-1} \log \|\mathsf{S}^{[\ell]}\|_1 - \theta_1 \right| \leq \theta_1 \eta / 10, \quad \text{for all } \ell \geq \ell_\eta.$$

Then

$$ s_{\max}^{(\ell)} \leq \|\mathsf{S}^{[\ell]}\|_1 \leq e^{\theta_1 \ell(1+\eta/10)} \leq e^{2\theta_1 \ell} < R^{1/2}, $$

for $\ell \geq \ell_\eta$ and $R \geq e^{6\theta_1 \ell}$. Taking $O_Q(1) \geq 3$, we thus guarantee that the first term in the right-hand side of (7.1), equal to $O_Q(1) \cdot \|f^{(\ell)}\|_\infty R^{1/2}$, dominates the combined error. Thus it suffices to consider the case of (3.30). Since $R' \leq R$ and $|R - R'| \leq s_{\max}^{(\ell)} < R^{1/2}$, and $(R - R^{1/2}) \geq R^{3/4}$ for $R \geq 9$, the proposition will follow from the following lemma. $\square$

Lemma 7.3. *Suppose that the assumptions of Proposition 7.1 are satisfied. Then for $\mathbb{P}$-a.e. $\mathbf{a} \in \Omega$, for any $\eta \in (0,1)$, there exists $\ell_\eta = \ell_\eta(\mathbf{a}) \in \mathbb{N}$, such that for all $\ell \geq \ell_\eta$ and any bounded cylindrical function $f^{(\ell)}$ of level ℓ, for any $\overline{\mathfrak{e}}' \in X_+(\mathbf{a})$ such that $\mathfrak{h}(\overline{\mathfrak{e}}') = \zeta^{[\ell]}(x^{[\ell]})$, with $x^{(\ell)} \in \mathcal{Y}^{(\ell)}$ and $\omega \in \mathbb{R}$,*

$$(7.6) \qquad |S_R^{(\overline{\mathfrak{e}}',0)}(f^{(\ell)}, \omega)| \leq O_Q(1) \cdot \|f^{(\ell)}\|_\infty R^{1+\eta} \prod_{\ell+1 \leq k < \frac{\log R}{3\theta_1}} \left(1 - c_1 \cdot \max_{v \in GR(\zeta)} \left\| \omega |\zeta^{[k]}(v)|_{\vec{s}} \right\|_{\mathbb{R}/\mathbb{Z}}^2 \right),$$

whenever

$$R = \left| x^{(\ell)}[0, N-1] \right|_{\bar{s}^{(\ell)}} \geq e^{6\theta_1 \ell}.$$

Proof. Again we assume without loss of generality that $f^{(\ell)} = f_a^{(\ell)}$, where $f_a^{(\ell)}(\bar{\mathfrak{e}}, t) = \mathbb{1}_{\mathfrak{x}_a^{(\ell)}} \psi_a^{(\ell)}(t')$, with $t' \in [0, s_a^{(\ell)}]$ and $\psi_a^{(\ell)} \in C([0, s^{(\ell)}])$, see Section 3.5 for details. Recall the formula (3.30) which applies here:

$$(7.7) \qquad S_R^{(\bar{\mathfrak{e}}', 0)}(f_a^{(\ell)}, \omega) = \widehat{\psi}_a^{(\ell)}(\omega) \cdot \Phi_a^{\bar{s}^{(\ell)}}\left(x^{(\ell)}[0, N-1], \omega\right).$$

First observe that

$$(7.8) \qquad |\widehat{\psi}_a^{(\ell)}(\omega)| \leq \|\psi_a^{(\ell)}\|_1 \leq \|\psi_a^{(\ell)}\|_\infty s_a^{(\ell)} \leq \|f_a^{(\ell)}\|_\infty s_{\max}^{(\ell)}.$$

Next we apply (3.34), which we copy here for convenience:

$$(7.9) \quad \left| \Phi_a^{\bar{s}^{(\ell)}}\left(x^{(\ell)}[0, N-1], \omega\right) \right| \leq 2 \sum_{j=\ell}^{n} \|\mathsf{S}^{[\ell+1,j]}\|_1 \cdot \|\mathsf{S}_{j+1}\|_1 \cdot \prod_{\ell+1 \leq k \leq j-1} \left(1 - c_1 \cdot \max_{v \in GR(\zeta)} \left\| \omega |\zeta^{[k]}(v)|_{\bar{s}} \right\|_{\mathbb{R}/\mathbb{Z}}^2 \right),$$

where

$$(7.10) \qquad \min_{b \in \mathcal{A}} |\zeta^{[\ell+1,n]}(b)| \leq N \leq 2 \max_{b \in \mathcal{A}} |\zeta^{[\ell+1,n+1]}(b)|.$$

By (4.2), we can assume that

$$(7.11) \qquad \|\mathsf{S}_{j+1}\|_1 \leq e^{j\theta_1(\eta/10)} \quad \text{for all} \quad j \geq \ell_\eta.$$

Further, note that

$$(7.12) \qquad \|\mathsf{S}^{[\ell+1,j+1]}\|_1 = \|\mathsf{S}^{[\ell+1,j]} Q \mathsf{S}_{\xi_{j+1}} Q\|_1 \geq 2\|\mathsf{S}^{[\ell+1,j]}\|_1,$$

since Q has strictly positive entries, and hence all entries of $Q\mathsf{S}_{\xi_{j+1}}Q$ are not less than $m \geq 2$.

From (7.11) and (7.12), recalling that $c_1 \leq 1/4$, we obtain that the sum in (7.9) is bounded above by $O_Q(1)$ times the last, n-th term, yielding

$$(7.13)$$
$$\left| \Phi_a^{\bar{s}^{(\ell)}}\left(x^{(\ell)}[0, N-1], \omega\right) \right| < O_Q(1) \cdot \|\mathsf{S}^{[\ell+1,n]}\|_1 \cdot e^{n\theta_1(\eta/10)} \cdot \prod_{\ell+1 \leq k \leq n-1} \left(1 - c_1 \cdot \max_{v \in GR(\zeta)} \left\| \omega |\zeta^{[k]}(v)|_{\bar{s}} \right\|_{\mathbb{R}/\mathbb{Z}}^2 \right).$$

This, together with (7.7), (7.8), is already very close to the desired (7.6), but we need to relate N, n, and R. First observe that

$$(7.14) \qquad R = \left| x^{(\ell)}[0, N-1] \right|_{\bar{s}^{(\ell)}} \in [N s_{\min}^{(\ell)}, N s_{\max}^{(\ell)}].$$

Note that

$$(7.15) \qquad s_{\max}^{(\ell)} \leq \operatorname{col}(Q^t) \cdot s_{\min}^{(\ell)},$$

since $s_{\max}^{(\ell)}$ and $s_{\max}^{(\ell)}$ are respectively the maximal and minimal components of $\vec{s}^{(\ell)} = (\mathsf{S}^{[\ell]})^t \vec{s} = Q^t (\mathsf{S}^{[\ell-1]})^t \vec{s}$ (recall that $\mathsf{S}_\ell = Q$ by assumption). Further, by (7.10) and (3.17),

$$N \geq (\operatorname{col}(Q^t))^{-1} \cdot \|\mathsf{S}^{[\ell+1,n]}\|_1,$$

since $|\zeta^{[\ell+1,n]}(b)|$ is a column sum of $\mathsf{S}^{[\ell+1,n]}$, a matrix which starts and ends with Q. Therefore,

$$(7.16) \qquad R \geq N s_{\min}^{(\ell)} \geq (\operatorname{col}(Q^t))^{-2} \cdot \|\mathsf{S}^{[\ell+1,n]}\|_1 \cdot s_{\max}^{(\ell)}.$$

Comparing (7.7), (7.8), and (7.13), we see that in order to conclude the proof of (7.6), it remains to show, first, that

$$(7.17) \qquad R^\eta \geq e^{n\theta_1(\eta/10)},$$

and second, that

$$(7.18) \qquad n \geq (\log R)/(3\theta_1).$$

Since $s_{\max}^{(\ell)} \geq (\operatorname{col}(Q^t))^{-1}\|\mathsf{S}^{[\ell]}\|_1$, we obtain from (7.16) and (7.5) that

$$R \geq (\operatorname{col}(Q^t))^{-3} \cdot \|\mathsf{S}^{[\ell+1,n]}\|_1 \cdot |\mathsf{S}^{[\ell]}\|_1 \geq (\operatorname{col}(Q^t))^{-3} \cdot \|\mathsf{S}^{[n]}\|_1 \geq (\operatorname{col}(Q^t))^{-3} \cdot e^{\theta_1 n(1-\eta/10)},$$

confirming (7.17) once ℓ, and hence n is sufficiently large. On the other hand, by the upper bound in (7.10) and (7.14),

$$R \leq N s_{\max}^{(\ell)} \leq 2\|\mathsf{S}^{[\ell]}\|_1 \cdot \|\mathsf{S}^{[\ell+1,n]}\|_1.$$

Since all matrices involved begin and end with Q, it is easy to see that

$$\|\mathsf{S}^{[\ell]}\|_1 \cdot \|\mathsf{S}^{[\ell+1,n]}\|_1 \leq \operatorname{col}(Q^t) \cdot \|\mathsf{S}^{[\ell]} \cdot \mathsf{S}^{[\ell+1,n]}\|_1 = \operatorname{col}(Q^t) \cdot \|\mathsf{S}^{[n]}\|_1,$$

and then (7.5) yields that

$$R \leq O_Q(1) \cdot e^{\theta_1 n(1+\eta/10)},$$

which certainly guarantees (7.18), once ℓ, and hence n, is sufficiently large. Now the lemma and the proposition are proved completely. $\qquad \square$

8. Linear algebra and the choice of good return words

In this section, as well as the next one, we continue to work with a $\mathbb{P}$-generic 2-sided sequence $\mathbf{a} \in \Omega_{\mathbf{q}}$. In view of the assumption (a′) of Theorem 5.1, we can fix unit basis vectors $\vec{e}_j^{(n)}$, $j = 1, 2$, for the one-dimensional subspaces $E_{\sigma^n \mathbf{a}}^j$, $j = 1, 2$, $n \geq 0$, such that

$$(8.1) \qquad \mathbb{A}(n, \mathbf{a})\vec{e}_j^{(0)} = A(n, j)\vec{e}_j^{(n)} \quad \text{for some} \quad A(n, j) > 0.$$

By (ii) in Oseledets Theorem, we have $\frac{1}{n} \log A(n, j) \to \theta_j$, $j = 1, 2$.

We start with an observation about the Lyapunov-Oseledets basis $\{\vec{e}_j^{(n)}\}_{j=1}^2$ of the unstable subspace. All the matrices $\mathbb{A}(n, \mathbf{a})$ are non-negative. Thus,

$$\|\mathbb{A}(n, \mathbf{a})\vec{x}\| \leq \|\mathbb{A}(n, \mathbf{a})\vec{x}^{|\cdot|}\|,$$

hence $\vec{e}_1^{(n)} \in \mathbb{R}_+^m$ (the positive cone) for all $n \geq 0$, and so

$$(8.2) \qquad \vec{e}_1^{(n)} \in A(\mathbf{a}_n)\mathbb{R}_+^m = Q^t A(\mathbf{p}_n)Q^t\mathbb{R}_+^m \subset Q^t\mathbb{R}_+^m.$$

On the other hand, under our assumptions the image of the positive cone $\mathbb{A}(n, \mathbf{a})\mathbb{R}_+^m$ shrinks to a single direction exponentially fast. (The fact that the cone shrinks to a single direction is equivalent to unique ergodicity, see (2.2) and Veech [32, 33].) It follows that the basis vector $\vec{e}_2^{(n)}$ does not lie in $\mathbb{R}_+^m$ for all n, otherwise we would get a contradiction with (iii) in Oseledets Theorem. Combined with (8.2), this implies that the angle between $\vec{e}_1^{(n)}$ and $\vec{e}_2^{(n)}$ is bounded away from zero by a constant depending only on Q.

We will next need an elementary fact from linear algebra.

Lemma 8.1. *Let $\mathcal{B} = \{\vec{x}_j\}_{j \leq m}$ be a basis of $\mathbb{R}^m$, and let $\{\vec{\xi}_1, \ldots, \vec{\xi}_r\} \subset \mathbb{R}^m$ be a linearly independent set, with $r \leq m$. Then there exists a subset $\{x_i\}_{i \in I} \subset \mathcal{B}$ of cardinality r such that*

$$|D_I| := \left| \det\left(\langle \vec{x}_i, \vec{\xi}_j \rangle \right)_{i \in I, j \leq r} \right| \geq C_{\mathcal{B}} \|\vec{\xi}_1 \wedge \cdots \wedge \vec{\xi}_r\|,$$

where $C_{\mathcal{B}}$ depends only on the basis $\mathcal{B}$.

Proof. Let $\mathcal{T}$ be the linear isomorphism which takes the standard basis $\{\vec{e}_j\}_{j \leq m}$ of $\mathbb{R}^m$ into $\mathcal{B}$. Then

$$D_I = \det\left(\langle \vec{x}_i, \vec{\xi}_j \rangle \right)_{i \in I, j \leq r} = \det\left(\langle \mathcal{T}\vec{e}_i, \vec{\xi}_j \rangle \right)_{i \in I, j \leq r} = \det\left(\langle \vec{e}_i, \mathcal{T}^*\vec{\xi}_j \rangle \right)_{i \in I, j \leq r}$$

The latter determinant is the order-r minor of the matrix whose columns are $\mathcal{T}^*\vec{\xi}_j$, $j = 1, \ldots, r$, corresponding to the rows indexed by I. Thus,

$$\sum_{\#I=r} |D_I|^2 = \|\mathcal{T}^*\vec{\xi}_1 \wedge \cdots \wedge \mathcal{T}^*\vec{\xi}_r\|^2 \geq \left\| (\textstyle\bigwedge^r \mathcal{T}^*)^{-1} \right\|^{-1} \|\vec{\xi}_1 \wedge \cdots \wedge \vec{\xi}_r\|^2.$$

We are using here that $\mathcal{T}^*$ is invertible, hence its exterior power is invertible. Thus,

$$\max\{|D_I| : \#I = r\} \geq \left\| (\textstyle\bigwedge^r \mathcal{T}^*)^{-1} \right\|^{-1/2} \binom{m}{r}^{-1/2} \|\vec{\xi}_1 \wedge \cdots \wedge \vec{\xi}_r\|,$$

and the proof is complete. $\qquad\qquad\square$

We now return to our theorem, in which $r = 2$. Let $\{u_j\}_{j \leq m}$ be the good return words from the Assumption (c') of Theorem 5.1. We will choose a sequence of words $v_n \in \{u_j\}_{j \leq m}$, depending on our generic $\mathbf{a} \in \Omega$. For $n \geq 1$ consider

$$(8.3) \qquad \boldsymbol{\Theta}_n := \begin{pmatrix} A(n,1)\langle \vec{\ell}(v_n), \vec{e}_1^{(n)} \rangle & A(n,2)\langle \vec{\ell}(v_n), \vec{e}_2^{(n)} \rangle \\ A(n+1,1)\langle \vec{\ell}(v_{n+1}), \vec{e}_1^{(n+1)} \rangle & A(n+1,2)\langle \vec{\ell}(v_{n+1}), \vec{e}_2^{(n+1)} \rangle \end{pmatrix}.$$

Below C_ζ denotes a constant depending only on the substitution $\zeta = \zeta(\mathbf{q})$.

Lemma 8.2. *For $\mathbb{P}$-a.e. $\mathbf{a} \in \Omega$ we can choose the words $v_n \in \{u_j\}_{j \leq m}$, so that for all $n \geq 1$,*

$$(8.4) \qquad \|\mathbf{\Theta}_n^{-1}\|_\infty \leq C_\zeta \cdot \frac{\max\{A(n+j,i); \ j=0,1; i=1,2\}}{A(n,1)A(n+1,2)}$$

and

$$(8.5) \qquad \|\mathbf{\Theta}_{n+1}\mathbf{\Theta}_n^{-1}\|_\infty \leq C_\zeta \cdot \frac{\max_{j=0,1,2} A(n+j,1)}{A(n,1)} \cdot \frac{\max_{j=0,1,2} A(n+j,2)}{A(n,2)}.$$

Proof. We are going to choose v_n inductively. Pick v_1 arbitrarily, and suppose $v_1, \ldots, v_n$ have been chosen. For $i \leq m$ consider

$$\Delta_i := \det \begin{pmatrix} A(n,1)\langle \vec{\ell}(v_n), \vec{e}_1^{(n)} \rangle & A(n,2)\langle \vec{\ell}(v_n), \vec{e}_2^{(n)} \rangle \\ A(n+1,1)\langle \vec{\ell}(u_i), \vec{e}_1^{(n+1)} \rangle & A(n+1,2)\langle \vec{\ell}(u_i), \vec{e}_2^{(n+1)} \rangle \end{pmatrix}.$$

Observe that

$$(8.6) \qquad \det \begin{pmatrix} \langle \vec{\ell}(u_i), \vec{e}_1^{(n+1)} \rangle & \Delta_i \\ \langle \vec{\ell}(u_j), \vec{e}_1^{(n+1)} \rangle & \Delta_j \end{pmatrix} = A(n,1)A(n+1,2)\langle \vec{\ell}(v_n), \vec{e}_1^{(n)} \rangle D_{ij},$$

where

$$D_{ij} := \det \begin{pmatrix} \langle \vec{\ell}(u_i), \vec{e}_1^{(n+1)} \rangle & \langle \vec{\ell}(u_i), \vec{e}_2^{(n+1)} \rangle \\ \langle \vec{\ell}(u_j), \vec{e}_1^{(n+1)} \rangle & \langle \vec{\ell}(u_j), \vec{e}_2^{(n+1)} \rangle \end{pmatrix}.$$

Note that $\vec{\xi}_1 := \vec{e}_1^{(n+1)} \in Q^t \mathbb{R}_+^m$ and $\vec{\xi}_2 := \vec{e}_2^{(n+1)} \notin \mathbb{R}_+^m$ by the comments above. Thus, the angle between $\vec{\xi}_1$ and $\vec{\xi}_2$ is bounded away from zero, uniformly in n. Hence we can apply Lemma 8.1 to these vectors and find $i \neq j$ such that

$$|D_{ij}| \geq c_3 > 0,$$

independent of n. Note that for all $i \leq m$,

$$(8.7) \qquad 0 < c_4 \leq |\langle \vec{\ell}(u_i), \vec{e}_1^{(n)} \rangle| \leq C_5 := \max_{i \leq m} \|\vec{\ell}(u_i)\|_2 < \infty$$

for some positive constant $c_4 = c_4(Q)$ independent of n, since $\vec{\ell}(u_i)$, $i \leq m$, are positive integer vectors. It follows from (8.6) and (8.7) that

$$\max_i |\Delta_i| \geq \frac{c_3 c_4}{2C_5} A(n,1)A(n+1,2).$$

We choose $v_{n+1} \in \{u_i\}_{i \leq m}$ to maximize $|\Delta_i|$. Denote $\Delta^{(n)} = \det(\mathbf{\Theta}_n)$. As a result, we will have for all $n \geq 1$:

$$(8.8) \qquad |\Delta^{(n)}| \geq \frac{c_3 c_4}{2C_5} A(n,1)A(n+1,2),$$

which implies (8.4). Also, a direct calculation, combined with (8.7) and (8.8), yields (8.5). $\qquad \square$

Corollary 8.3. *For $\mathbb{P}$-a.e. $\mathbf{a} \in \Omega_{\mathbf{q}}$ we can choose the words $v_n \in \{u_j\}_{j \leq m}$, so that for any $\delta_1 > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,*

$$(8.9) \qquad \|\mathbf{\Theta}_n^{-1}\|_{\infty} \leq C_{\zeta} \exp[-(\theta_2 - \delta_1)n]$$

and

$$(8.10) \qquad \|\mathbf{\Theta}_{n+1}\mathbf{\Theta}_n^{-1}\|_{\infty} \leq C_{\zeta} \exp[2(W_n + W_{n+1})],$$

where C_{ζ} is the constant from Lemma 8.2 and W_n are defined in (6.1).

Proof. This is a combination of the last lemma and Oseledets Theorem. First we prove (8.9). By Oseledets Theorem, for $\mathbb{P}$-a.e. $\mathbf{a} \in \Omega_{\mathbf{q}}$, for all n sufficiently large,

$$\exp[(\theta_i - \delta_1/4)n] \leq A(n, i) \leq \exp[(\theta_i + \delta_1/4)n], \quad i = 1, 2.$$

We will use (8.4), where clearly the maximum in the numerator is (eventually) attained for $i = 1$, to obtain for n sufficiently large:

$$\begin{aligned}
\|\mathbf{\Theta}_n^{-1}\|_{\infty} &\leq C_{\zeta} \exp[(\theta_1 + \delta_1/4)n + 1 - (\theta_1 - \delta_1/4)n - (\theta_2 - \delta_1/4)n + 1] \\
&= C_{\zeta} \exp[(\theta_1 - \theta_2 + \delta_1/2) - (\theta_2 - 3\delta_1/4)n].
\end{aligned}$$

For n sufficiently large,

$$\theta_1 - \theta_2 + \delta_1/2 \leq (\delta_1/4)n,$$

and (8.9) follows. Next, let us verify (8.10). Equation (8.1) implies that

$$A(n+1, j)\vec{e}_j^{(n+1)} = A(\mathbf{a}_n)A(n, j)\vec{e}_j^{(n)}, \quad j = 1, 2,$$

hence $A(n+1, j)/A(n, j) \leq e^{W_n}$ by (6.1). Now (8.10) follows from (8.5). $\qquad \square$

9. Beginning of the proof of Theorem 5.1

Now we proceed with the proof of Theorem 5.1. As before, we fix a $\mathbb{P}$-generic point $\mathbf{a} \in \Omega_{\mathbf{q}}$. Under the assumptions of Theorem 5.1,

$$(9.1) \qquad \vec{s} = \sum_{j=1}^{2} a_j \vec{e}_j^{(0)} + P_{\mathbf{a}}^{st}\vec{s},$$

where $a_j = P_{\mathbf{a}}^j(\vec{s})$ and $P_{\mathbf{a}}^{st}$ is the projection to the stable subspace $E_{\mathbf{a}}^3 \oplus \cdots \oplus E_{\mathbf{a}}^r$ in (4.4). Recall that $\vec{s} \in \Delta_m := \{\vec{s} \in \mathbb{R}_+^m : \|\vec{s}\|_1 = 1\}$. Our goal is to prove that for all $\beta > 0$ and $B > 1$, for a.e. $\vec{s}$, with $\min_{j=1,2} |a_j| \geq \beta$,

$$(9.2) \qquad \sigma_f(B(\omega, r)) \leq C(\mathbf{a}, \|f\|_L) \cdot r^{\gamma} \quad \text{for } \omega \in [B^{-1}, B] \text{ and } 0 < r \leq r_0(\mathbf{a}, \beta, B).$$

Fix $\beta > 0$ and $B > 1$ for the rest of the proof. Recall that dependence on $\vec{s}$ in the estimate is "hidden" in σ_f, which is the spectral measure of the suspension flow corresponding to the roof function given by $\vec{s}$.

Fix the sequence of good return words v_n from Lemma 8.2. For $n \in \mathbb{N}$ and $\omega \in [B^{-1}, B]$, let

(9.3) $$\omega|\zeta^{[n]}(v_n)|_{\vec{s}} = K_n + \varepsilon_n, \quad \text{where } K_n \in \mathbb{N}, \ |\varepsilon_n| \leq 1/2,$$

so that

$$\||\omega|\zeta^{[n]}(v_n)|_{\vec{s}}\|_{\mathbb{R}/\mathbb{Z}} = |\varepsilon_n|.$$

We should keep in mind that K_n and ε_n depend on ω and on $\vec{s}$, although this is suppressed in notation to avoid clutter. Given $\beta, \varrho, \delta > 0$ and $B > 1$, define

$$E_N(\varrho, \delta, \beta, B) \quad := \quad \left\{ \vec{s} \in \Delta_m : \ \min_{j=1,2} |P_{\mathbf{a}}^j(\vec{s})| \geq \beta \text{ and } \exists \omega \in [B^{-1}, B] \right.$$

$$\left. \text{such that } \operatorname{card}\{n \leq N : |\varepsilon_n| \geq \varrho\} < \delta N \right\}.$$

and

$$\mathcal{E}(\varrho, \delta, \beta, B) := \bigcap_{N_0=1}^{\infty} \bigcup_{N=N_0}^{\infty} E_N(\varrho, \delta, \beta, B).$$

Proposition 9.1. *There exist* $\varrho > 0$ *such that for* $\mathbb{P}$-*a.e.* $\mathbf{a} \in \Omega_{\mathbf{q}}$ *we have*

$$\forall \epsilon > 0, \ \exists \delta_0 > 0, \ \forall \beta > 0, \ \forall B > 1 : \quad \delta < \delta_0 \implies \dim_H(\mathcal{E}(\varrho, \delta, \beta, B)) < m - 2 + \epsilon.$$

In the remaining part of this section, we derive Theorem 5.1 from Proposition 9.1. Then in the next section we use the "Erdős-Kahane argument" to prove the proposition.

Proof of Theorem 5.1 assuming Proposition 9.1. In view of Lemma 3.1, it suffices to show

(9.4) $$|S_R^{(\overline{\mathfrak{e}},t)}(f, \omega)| \leq \widetilde{C}(\mathbf{a}, \|f\|_L) \cdot R^{1-\gamma/2}, \quad \text{for } R \geq R_0(\mathbf{a}, \beta, B),$$

for some $\gamma \in (0, 1)$, uniformly in $(\overline{\mathfrak{e}}, t) \in \mathfrak{X}^{\vec{s}}$. We will specify γ at the end of the proof, see (9.10).

Since f is weakly Lipschitz on $\mathfrak{X}^{\vec{s}}$ (see Section 2.1), for almost every $\mathbf{a}$ we can approximate f, for any $\ell \in \mathbb{N}$, by a function $f^{(\ell)}$, which is cylindrical of level ℓ, and has sup-norm bounded by $\|f\|_{\infty}$, so that

$$\|f - f^{(\ell)}\|_{\infty} \leq \|f\|_L \cdot \nu_+([\mathfrak{e}_1 \ldots \mathfrak{e}_n]).$$

We can do this simply taking $f^{(\ell)}(\overline{\mathfrak{e}}, t) := f(\overline{\mathfrak{e}}^{(\ell)}, t)$, where $\overline{\mathfrak{e}}^{(\ell)}$ agrees with $\overline{\mathfrak{e}}$ down to level ℓ after which it is extended to infinity in any fixed way. By (2.6), (2.3), and (2.2), we have

$$\lim_{n \to \infty} \frac{\log \nu_+([\mathfrak{e}_1 \ldots \mathfrak{e}_n])}{n} = -\theta_1,$$

$\mathbb{P}$-almost surely, hence for ℓ sufficiently large we have

(9.5) $$\|f - f^{(\ell)}\|_{\infty} \leq \|f\|_L \cdot e^{-\frac{1}{2}\theta_1 \ell}.$$

Recall that $S_R^{(\overline{\mathfrak{e}},t)}(f, \omega) = \int_0^R e^{-2\pi i \omega t} f \circ h_\tau(\overline{\mathfrak{e}}, t) \, d\tau$. Let

(9.6) $$\ell := \left\lfloor \frac{2\gamma \log R}{\theta_1} \right\rfloor.$$

Then (9.5) yields

$$|S_R^{(\bar{\mathbf{e}},t)}(f,\omega) - S_R^{(\bar{\mathbf{e}},t)}(f^{(\ell)},\omega)| \leq R \cdot \|f\|_L \cdot e^{-\frac{1}{2}\theta_1\ell} \leq e^{\theta_1/2} \cdot \|f\|_L \cdot R^{1-\gamma}.$$

Thus, it is enough to obtain (9.4), with f replaced by $f^{(\ell)}$. For the latter, we can apply Proposition 7.1. Recall the inequality (7.1), using the sequence of good return words $\{v_n\}$ and $\eta = \gamma/2$:

$$|S_R^{(\bar{\mathbf{e}},t)}(f^{(\ell)},\omega)| \leq O_Q(1) \cdot \|f^{(\ell)}\|_\infty \left(R^{1/2} + R^{1+\gamma/2} \prod_{\ell+1 \leq n < \frac{\log R}{4\theta_1}} \left(1 - c_1 \cdot \|\omega|\zeta^{[n]}(v_n)|_{\vec{s}}\|_{\mathbb{R}/\mathbb{Z}}^2\right) \right),$$

for $\ell \geq \ell_\gamma$ and all $R \geq e^{8\theta_1\ell}$. We can ensure that $\ell \geq \ell_\gamma$ by taking R_0 sufficiently large, and $R \geq e^{8\theta_1\ell}$ will follow from (9.6) if $\gamma \leq 1/16$. Since our goal is (9.4), we can discard the $R^{1/2}$ term immediately. Now choose $\varrho > 0$ and $\delta > 0$ from Proposition 9.1 such that $\dim_H(\mathcal{E}(\varrho,\delta,\beta,B)) < m-1 = \dim(\Delta_m)$ for all $\beta > 0, B > 1$. It is enough to verify

$$(9.7) \qquad \prod_{\ell+1 \leq n < \frac{\log R}{4\theta_1}} \left(1 - c_1 \cdot \|\omega|\zeta^{[n]}(v_n)|_{\vec{s}}\|_{\mathbb{R}/\mathbb{Z}}^2\right) \leq O_{\mathbf{a},\|f\|_L}(1) \cdot R^{-\gamma}, \quad \omega \in [B^{-1}, B],$$

for $R \geq R_0(\mathbf{a},\beta,B)$, for all vectors $\vec{s} \in \Delta_m \setminus \mathcal{E}(\varrho,\delta,\beta,B)$, for which $\min_{j=1,2} |P_{\mathbf{a}}^j(\vec{s})| \geq \beta$, thus obtaining an even stronger than 'almost every $\vec{s}$' statement.

By definition, $\vec{s} \notin \mathcal{E}(\varrho,\delta,\beta,B)$ means that there exists $N_0 = N_0(\mathbf{a},\beta,B) \in \mathbb{N}$ such that $\vec{s} \notin E_N(\varrho,\delta,\beta,B)$ for all $N \geq N_0$. Let

$$(9.8) \qquad N = \left\lfloor \frac{\log R}{4\theta_1} \right\rfloor \quad \text{and} \quad R_0 = e^{4\theta_1(N_0+1)}.$$

Then $R \geq R_0$ implies $N \geq N_0$, and the product in (9.7) is less than or equal to

$$\prod_{n=\ell+1}^{N-1} (1 - c_1|\varepsilon_n|^2),$$

where we also use (9.3). By definition, $\vec{s} \notin E_N(\varrho,\delta,\beta,B)$ means that there are at least $\lceil \delta N \rceil$ numbers $n \in \{1,\dots,N\}$ with $|\varepsilon_{k_n}| \geq \varrho$, hence the left-hand side of (9.7) is bounded above by $(1 - c_1\varrho^2)^{\delta N-1-\ell}$. Recalling (9.8) and (9.6), we see that

$$(9.9) \qquad (1 - c_1\varrho^2)^{\delta N-1-\ell} \leq O(1) \cdot (1 - c_1\varrho^2)^{(\delta \log R)/(8\theta_1)} \leq O(1) \cdot R^{-\gamma},$$

provided that

$$(9.10) \qquad \gamma \leq \min\left\{ \frac{\delta}{16}, \frac{-\delta \log(1 - c_1\varrho^2)}{8\theta_1} \right\},$$

with the two conditions for γ needed for the left and right inequality in (9.9) correspondingly. Now the proof of (9.7) is complete, and it remains to verify Proposition 9.1 to conclude the proof of Theorem 5.1. $\qquad\square$

10. The Erdős-Kahane method and the conclusion of the proof of Theorem 5.1

In this section, we prove Proposition 9.1. We need some preparation first. Fix $\beta \in (0,1)$ and suppose that $\vec{s} \in \Delta_m$ is such that $\min_{j=1,2} |P_{\mathbf{a}}^j(\vec{s})| \geq \beta$. Further, let $B > 1$ and $\omega \in [B^{-1}, B]$. Recall (9.1) and (9.3). In view of (8.1), we have for $n \geq 1$, denoting $\xi_n = \langle \vec{\ell}(v_n), (\mathsf{S}^{[n]})^t P_{\mathbf{a}}^{st} \vec{s} \rangle$:

$$|\zeta^{[n]}(v_n)|_{\vec{s}} = \langle \vec{\ell}(\zeta^{[n]}(v_n)), \vec{s} \rangle = \langle S^{[n]} \vec{\ell}(v_n), \vec{s} \rangle = \langle \vec{\ell}(v_n), (\mathsf{S}^{[n]})^t \vec{e}_j^{(0)} \rangle$$

$$(10.1) \qquad = \sum_{j=1}^{2} a_j A(n,j) \langle \vec{\ell}(v_n), \vec{e}_j^{(n)} \rangle + \xi_n.$$

By the Assumption (a′) of Theorem 5.1, for $\mathbb{P}$-a.e. $\mathbf{a} \in \Omega_{\mathbf{q}}$, we have $\limsup_{n \to \infty} n^{-1} \log \|(\mathsf{S}^{[n]})^t P_{\mathbf{a}}^{st}\| \leq \theta_3 < 0$, hence

$$(10.2) \qquad \limsup_{n \to \infty} n^{-1} \log |\xi_n| \leq \theta_3 < 0.$$

Now let $\vec{s} \in \Delta_m$ and $\omega \in [B^{-1}, B]$ be from the definition of $E_N(\varrho, \delta, \beta, B)$. Recall (9.3), and denote

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \vec{K}_n = \begin{pmatrix} K_n \\ K_{n+1} \end{pmatrix}, \quad \vec{\varepsilon}_n = \begin{pmatrix} \varepsilon_n \\ \varepsilon_{n+1} \end{pmatrix}, \quad \vec{\xi}_n = \begin{pmatrix} \xi_n \\ \xi_{n+1} \end{pmatrix}.$$

We need the matrices $\mathbf{\Theta}_n$ defined in (8.3):

$$\mathbf{\Theta}_n = \begin{pmatrix} A(n,1) \langle \vec{\ell}(v_n), \vec{e}_1^{(n)} \rangle & A(n,2) \langle \vec{\ell}(v_n), \vec{e}_2^{(n)} \rangle \\ A(n+1,1) \langle \vec{\ell}(v_{n+1}), \vec{e}_1^{(n+1)} \rangle & A(n+1,2) \langle \vec{\ell}(v_{n+1}), \vec{e}_2^{(n+1)} \rangle \end{pmatrix}.$$

The equations (9.3) for $n, n+1$, in view of (10.1), combine into

$$\omega \mathbf{\Theta}_n \vec{a} = \vec{K}_n + \vec{\varepsilon}_n - \omega \vec{\xi}_n,$$

hence

$$(10.3) \qquad \vec{a} = \omega^{-1} \mathbf{\Theta}_n^{-1} (\vec{K}_n + \vec{\varepsilon}_n - \omega \vec{\xi}_n).$$

It follows that

$$(10.4) \qquad a_j = \omega^{-1} [\mathbf{\Theta}_n^{-1} (\vec{K}_n + \vec{\varepsilon}_n - \omega \vec{\xi}_n)]_j, \quad j = 1, 2,$$

where $[\cdot]_j$ denotes j-th component of a vector. Observe that

$$(10.5) \qquad 0 < \beta \leq |a_1|, |a_2| \leq C_{\mathbf{a}},$$

where the upper bound comes from the fact that $\|\vec{s}\|_1 = 1$ and the angles between Lyapunov subspaces at $\mathbf{a}$ depend on $\mathbf{a}$. Choose $\delta_1 > 0$ such that $\theta_2 - \delta_1 > 0$ and $\theta_3 + \delta_1 < 0$. Note that $\|\vec{\varepsilon}_n\|_\infty \leq \frac{1}{2}$ for all n, and for $n \geq n_0(\mathbf{a})$,

$$(10.6) \qquad |\xi_{k_n}| \leq e^{(\theta_3 + \delta_1)k_n} \leq e^{(\theta_3 + \delta_1)n}.$$

Since $|\omega| \leq B$, we have

$$\|\omega \vec{\xi}_n\|_\infty \leq 1/2 \quad \text{for } n \geq n_0(\mathbf{a}) + C_{\mathrm{Lyap}} \cdot \log B, \quad \text{where } C_{\mathrm{Lyap}} = |\theta_3 + \delta_1|^{-1}.$$

Here and below we denote by $n_0(\mathbf{a})$ a generic integer constant depending on $\mathbf{a}$, and possibly also on the Lyapunov spectrum (below it will also depend on β), and by C_{Lyap} a constant which depends only on the Lyapunov spectrum. Similarly, $C_{\mathbf{a}} \geq 1$ is a generic constant depending on $\mathbf{a}$, and possibly also on the Lyapunov spectrum. These constants may be different from line to line.

Thus, $\|\vec{\varepsilon}_n - \omega\vec{\xi}_n\|_\infty \leq 1$ for large enough n, so (10.4), the lower bound for ω, and (8.9) yield for $j = 1, 2$:

$$
\begin{aligned}
|a_j - \omega^{-1}(\mathbf{\Theta}_n^{-1}\vec{K}_n)_j| &\leq B\|\mathbf{\Theta}_n^{-1}\|_\infty \\
(10.7) \qquad\qquad &\leq BC_\zeta e^{-(\theta_2 - \delta_1)k_n} \leq BC_\zeta e^{-(\theta_2 - \delta_1)n}, \quad \text{for } n \geq n_0(\mathbf{a}).
\end{aligned}
$$

In view of (10.5) and (10.7), we obtain for $j = 1, 2$:

$$0 < \beta/2 \leq |\omega^{-1}(\mathbf{\Theta}_n^{-1}\vec{K}_n)_j| \leq 2C_{\mathbf{a}}, \quad \text{for } n \geq n_0(\mathbf{a}, \beta) + C_{\mathrm{Lyap}} \cdot \log B.$$

From these bounds and (10.7), we obtain, again for $n \geq n_0(\mathbf{a}, \beta) + C_{\mathrm{Lyap}} \cdot \log B$:

$$
\begin{aligned}
\left| \frac{a_2}{a_1} - \frac{[\mathbf{\Theta}_n^{-1}\vec{K}_n]_2}{[\mathbf{\Theta}_n^{-1}\vec{K}_n]_1} \right| &\leq \frac{|a_2 - \omega^{-1}(\mathbf{\Theta}_n^{-1}\vec{K}_n)_2|}{|a_1|} + \frac{|\omega^{-1}(\mathbf{\Theta}_n^{-1}\vec{K}_n)_2| \cdot |a_1 - \omega^{-1}(\mathbf{\Theta}_n^{-1}\vec{K}_n)_1|}{|a_1| \cdot |\omega^{-1}(\mathbf{\Theta}_n^{-1}\vec{K}_n)_1|} \\
(10.8) \qquad\qquad &\leq 6C_1(\mathbf{a})\beta^{-2}BC_\zeta \exp[-(\theta_2 - \delta_1)n].
\end{aligned}
$$

On the other hand, comparing (10.3) for n and $n + 1$ yields

$$\vec{K}_{n+1} + \vec{\varepsilon}_{n+1} - \omega\vec{\xi}_{n+1} = \mathbf{\Theta}_{n+1}\mathbf{\Theta}_n^{-1}[\vec{K}_n + \vec{\varepsilon}_n - \omega\vec{\xi}_n],$$

hence, using $|\omega| \leq B$, we obtain

$$\|\vec{K}_{n+1} - \mathbf{\Theta}_{n+1}\mathbf{\Theta}_n^{-1}\vec{K}_n\|_\infty \leq \|\vec{\varepsilon}_{n+1}\|_\infty + B\|\vec{\xi}_{n+1}\|_\infty + \|\mathbf{\Theta}_{n+1}\mathbf{\Theta}_n^{-1}\|_\infty(\|\vec{\varepsilon}_n\|_\infty + B\|\vec{\xi}_n\|_\infty).$$

This implies, in view of (8.10), for $n \geq n_0(\mathbf{a})$:

$$
\begin{aligned}
\left| K_{n+2} - [\mathbf{\Theta}_{n+1}\mathbf{\Theta}_n^{-1}\vec{K}_n]_2 \right| &\leq (1 + C_\zeta \exp[2(W_n + W_{n+1})] \times \\
(10.9) \qquad\qquad &\times (\max\{|\varepsilon_n|, |\varepsilon_{n+1}|, |\varepsilon_{n+2}|\} + B\max\{|\xi_n|, |\xi_{n+1}|, |\xi_{n+2}|\}) \,.
\end{aligned}
$$

Let

$$(10.10) \qquad\qquad M_n := 1 + C_\zeta \exp[2(W_n + W_{n+1})] \quad \text{and} \quad \rho_n = \frac{1}{4M_n} \,.$$

Lemma 10.1. *For all $n \geq n_0(\mathbf{a}, \beta) + C_{\mathrm{Lyap}} \log B$, we have the following, independent of $\omega \in [B^{-1}, B]$ and $\vec{s} \in \Delta_m$, satisfying $|a_j| = |P_{\mathbf{a}}^j(\vec{s})| \geq \beta > 0$:*
   (i) *Given K_n, K_{n+1}, there are at most $2M_n + 1$ possibilities for the integer K_{n+2};*
   (ii) *if $\max\{|\varepsilon_n|, |\varepsilon_{n+1}|, |\varepsilon_{n+2}|\} < \rho_n$, then K_{n+2} is uniquely determined by K_n, K_{n+1}.*

Proof. For part (i), we just use that $\|\vec{\varepsilon}_n\|_\infty \leq \frac{1}{2}$ for all n and $\|\vec{\xi}_n\|_\infty \leq (2B)^{-1}$ for $n \geq n_0(\mathbf{a}) + C_{\mathrm{Lyap}} \log B$, and that the number of integer points in an interval of length $2M_n$ is at most $2M_n+1$.

For part (ii), we claim that K_{n+2} belongs to a neighborhood of radius less than $\frac{1}{2}$, centered at $[\mathbf{\Theta}_{n+1}\mathbf{\Theta}_n^{-1}\vec{K}_n]_2$, under the given assumptions, for n sufficiently large. We have $M_n\rho_n = 1/4$, so it remains to make sure that

$$\max\{|\xi_n|, |\xi_{n+1}|, |\xi_{n+2}|\} \leq \rho_n/B$$

for n sufficiently large. Note that for any $\widetilde{\delta} > 0$ we have, by (6.3):

$$\rho_n \geq e^{-\widetilde{\delta}n}, \quad \text{for } n \geq n_0(\mathbf{a}).$$

Taking $\widetilde{\delta} < |\theta_3 + \delta_1|$ and combining the last inequality with (10.6) implies the desired claim. $\square$

Proof of Proposition 9.1. Let $\widetilde{E}_N(\delta, \beta, B)$ be defined by

$$\widetilde{E}_N(\delta, \beta, B) \quad := \quad \Big\{\vec{s} \in \Delta_m : \ \min_{j=1,2} |P_{\mathbf{a}}^j(\vec{s})| \geq \beta \text{ and } \exists\, \omega \in [B^{-1}, B] \text{ such that}$$

$$\mathrm{card}\{n \leq N : \ \max\{|\varepsilon_n|, |\varepsilon_{n+1}|, |\varepsilon_{n+2}|\} \geq \rho_n\} < \delta N\Big\}.$$

First we claim that $\mathbb{P}$-almost surely,

(10.11) $$\widetilde{E}_N(\delta, \beta, B) \supset E_N(\varrho, \delta/6, \beta, B)$$

for $N \geq N_0(\mathbf{a})$, where

(10.12) $$\varrho = (1/4)(1 + C_\zeta e^{2K})^{-1}, \quad \text{with} \quad K = 25L_1 \log(1/\delta).$$

Here C_ζ is from Lemma 8.2 and L_1 is from Proposition 6.1. Suppose $\vec{s} \notin \widetilde{E}_N(\delta, \beta, B)$. Then for all $\omega \in [B^{-1}, B]$ there exists a subset $\Gamma_N \subset \{1, \ldots, N\}$ of cardinality $\geq \delta N/3$ such that

$$|\varepsilon_{k_n}| \geq \rho_n \quad \text{for all } k_n \in \Gamma_N.$$

Observe that there are fewer than $\delta N/6$ integers $n \leq N$ for which $W_n + W_{n+1} > K$, for $N \geq N_0(\mathbf{a})$. Indeed, otherwise we can find $\Psi \subset \{1, \ldots, N\}$, with $|\Psi| \geq \delta N/12$, such that $W_n > K/2$ for $n \in \Psi$, hence

$$\sum\{W_n : \ n \in \Psi\} \geq K\delta N/24,$$

which contradicts (6.2) for $K > 24L_1 \log(1/\delta)$. In view of (10.10) and (10.12), it follows that

$$\mathrm{card}\big\{n \in \Gamma_N : \ \rho_n \geq \varrho\big\} \geq \delta N/6.$$

Thus $\vec{s} \notin E_N(\varrho, \delta/6, \beta, B)$ which confirms (10.11).

It follows that it is enough to estimate the dimension of

$$\widetilde{\mathcal{E}} := \widetilde{\mathcal{E}}(\delta, \beta, B) := \bigcap_{N_0=1}^{\infty} \bigcup_{N=N_0}^{\infty} \widetilde{E}_N(\delta, \beta, B).$$

Suppose $\vec{s} \in \widetilde{E}_N := \widetilde{E}_N(\delta, \beta, B)$; choose ω from the definition of $\widetilde{E}_N$, and find the corresponding sequence K_n, ε_n. In order to prove that $\dim_H(\widetilde{\mathcal{E}}) < m - 2 + \varepsilon$, it is enough to show that the set of a_2/a_1 corresponding to $\vec{s} \in \widetilde{\mathcal{E}}$ has Hausdorff dimension smaller than ε. We have from (10.8) that a_2/a_1 is covered by an interval of radius

$$(10.13) \qquad C_1(\mathbf{a}, \beta) \cdot B \exp[-(\theta_2 - \delta_1)N], \quad \text{for } N \geq N_0(\mathbf{a}, \beta) + C_{\text{Lyap}} \log B,$$

centered at $[\boldsymbol{\Theta}_n^{-1}\vec{K}_n]_2/[\boldsymbol{\Theta}_n^{-1}\vec{K}_n]_1$. Thus we need to estimate the number of sequences K_n, $n \leq N$, which may arise. Let Ψ_N be the set of $n \in \{1, \ldots, N\}$ for which we have $\max\{|\varepsilon_n|, |\varepsilon_{n+1}|, |\varepsilon_{n+2}|\} \geq \rho_n$. By the definition of $\widetilde{E}_N$ we have $|\Psi_N| < \delta N$. There are $\sum_{i < \delta N} \binom{N}{i}$ such sets. For a fixed Ψ_N the number of possible sequences $\{K_n\}$ is at most

$$\mathcal{B}_N := \prod_{n \in \Psi_N} (2M_n + 1),$$

times the number of "beginnings" $K_1, \ldots, K_{n_2}$, by Lemma 10.1. The number of possible "beginnings" is bounded, independent of N by a constant depending on β and B, in view of the a priori bounds on ω and $\vec{s}$. By the definition of M_n and (6.2), we have, for N sufficiently large:

$$\mathcal{B}_N \lesssim \exp\left(C'' \sum_{n \in \Psi_N} (W_n + W_{n+1})\right) \leq \exp\left[\widetilde{L} \log(1/\delta)(\delta N)\right].$$

Thus, by (10.13), the number of balls of radius $O_{\beta, B}(1)e^{-(\theta_2 - \delta_1)N}$ needed to cover $\widetilde{\mathcal{E}}$ is at most

$$(10.14) \quad O_{\beta, B}(1) \cdot \sum_{i < \delta N} \binom{N}{i} \exp\left[\widetilde{L} \log(1/\delta)(\delta N)\right] \leq O_{\beta, B}(1) \cdot \exp\left[(\widetilde{L} + C') \log(1/\delta)(\delta N)\right],$$

using (6.6) in the last inequality. Since $\delta \log(1/\delta) \to 0$ as $\delta \to 0$, we can choose $\delta_0 > 0$ so small that $\delta < \delta_0$ implies

$$\left[(\widetilde{L} + C') \log(1/\delta)(\delta N)\right] < \epsilon(\theta_2 - \delta_1)N,$$

whence $\widetilde{\mathcal{E}}$ has Hausdorff dimension less than ϵ, as desired. The proof of Proposition 9.1, and hence of Theorem 5.1, is complete.

$\square$

11. DERIVATION OF THEOREM 1.1 FROM THEOREM 4.1

Consider our surface M of genus 2. By the results of [9, Section 4] there is a correspondence between almost every translation flow and an element $\mathbf{a} \in \Omega$ (space of 2-sided Markov compacta), such that the (uniquely ergodic) flow is measure-theoretically conjugate to the uniquely ergodic flow $(X(\mathbf{a}), h_t^+)$ and hence to the suspension flow $(\mathfrak{X}^{\vec{s}}, h_t)$ over the Vershik map $(X_+(\mathbf{a}), \mathfrak{T})$ with the roof function corresponding to an appropriate vector $\vec{s}$, see Lemma 2.1. By construction (see [9]), this correspondence intertwines the Teichmüller flow on the space of Abelian differentials and a measure-preserving system $(\Omega, \mathbb{P}, \sigma)$, as considered at the beginning of our Section 4. The key

point here is that the Masur-Veech measure on the space of abelian differentials is taken, under this correpondence, to a measure mutually absolute continuous with the product of the measure $\mathbb{P}_+$ on Ω_+ and the Lebesgue measure on the 3-dimensional set of possible vectors $\vec{s}$ defining the suspension.

More precisely: as is well-known, the translation flow on the surface can be realized as a suspension flow over an interval exchange transformation (IET), see [40] for details. Veech [32] constructed, for any connected component of a stratum $\mathcal{H}$, a measurable finite-to-one map from the space $\mathcal{V}(\mathcal{R})$ of zippered rectangles corresponding to the Rauzy class $\mathcal{R}$, to $\mathcal{H}$, which intertwines the Teichmüller flow on $\mathcal{H}$ and a renormalization flow P_t that Veech defined on $\mathcal{V}(\mathcal{R})$. Observe that in our case the stratum $\mathcal{H}(2)$ is connected and corresponds to the Rauzy class of the IET with permutation $(4, 3, 2, 1)$. In general, the Veech mapping from $\mathcal{V}(\mathcal{R})$ to $\mathcal{H}$ is not bijective (it corresponds to passing from absolute to relative real cohomologies in the manifold M), but in our case the kernel is trivial, since the manifold has only one singularity and therefore there are no saddle connections. For background and complete details the reader is referred to [9] and [40].

Section 4.3 of [9] constructs the symbolic coding of the flow P_t on $\mathcal{V}(\mathcal{R})$, namely, a map

$$(11.1) \qquad\qquad \Xi_{\mathcal{R}} : (\mathcal{V}(\mathcal{R}), \widetilde{\mu}_2) \to (\Omega, \mathbb{P}),$$

defined almost everywhere, where $\widetilde{\mu}_2$ is the pull-back of the Masur-Veech measure μ_2 from $\mathcal{H}$ and $(\Omega, \mathbb{P})$ is a space of Markov compacta. The first return map of the flow P_t for an appropriate Poincaré section is mapped by $\Xi_{\mathcal{R}}$ to the shift map σ on $(\Omega, \mathbb{P})$. This correspondence maps the Rauzy-Veech cocycle over the Teichmüller flow into the renormalization cocycle for the Markov compacta. Moreover, the map $\Xi_{\mathcal{R}}$ induces a map defined for a.e. $\mathcal{X} \in \mathcal{V}(\mathcal{R})$, from the corresponding Riemann surface $M(\mathcal{X})$ to a Markov compactum $X(\mathbf{a}) \in \Omega$, intertwining their vertical and horizontal flows.

Now let f be a Lipschitz function on M with an abelian differential $\boldsymbol{\omega}$. Under the symbolic coding from [9], it is mapped into a weakly Lipschitz function on $X(\mathbf{a})$ and then to a weakly Lipschitz function $f^{\vec{s}}$ on $\mathfrak{X}^{\vec{s}}$ with $\vec{s}$ given by Lemma 2.1. By definition, its norm $\|f^{\vec{s}}\|_L$ is dominated by $\|f\|_L$ for all $\vec{s}$. Once we check all the assumptions, we can apply Theorem 4.1 and obtain the Hölder property of the spectrum of the suspension flow for $\mathbb{P}$-a.e. $\mathbf{a} \in \Omega$, for Lebesgue-a.e. $\vec{s}$, which in view of the mutual absolute continuity indicated above, is equivalent to the Hölder property for the flow h_t^+, as desired. Note that the dependence of r_0 on β (determined by $\mathbf{a}$ and $\vec{s}$) in Theorem 4.1 will be subsumed by the dependence of r_0 on $\mathbf{a}$ in Theorem 1.1.

In order to reduce Theorem 1.1 to Theorem 4.1, we must now check that the assumptions of Theorem 4.1 hold for the left shift on the space of Markov compacta endowed with the push-forward of the Masur-Veech smooth measure under the isomorphism of [9]. It is clear that condition (C1) follows from (b), which we discuss below. Condition (C2) holds because the renormalization matrices in the Rauzy-Veech induction all have determinant ± 1. Condition (C3) holds by a theorem of Zorich [41]. The condition (a) on the Lyapunov spectrum from Theorem 4.1

follows from results of Forni [18] in our case (later Avila and Viana [5] proved this for an arbitrary genus ≥ 2).

In order to ensure conditions (b) and (c), we need to recall the construction of the Markov compactum and Bratteli-Vershik realization of the translation flow from [9]. The symbolic representation of the translation flow on the 2-sided Markov compactum is obtained as the natural extension of the 1-sided symbolic representation for the IET which we now describe. An interval exchange is denoted by (λ, π), where π is the permutation of m subintervals and λ is the vector of their lengths. The well-known Rauzy induction (operations "a" and "b") proceeds by inducing on a smaller interval, so that the first return map is again an exchange of m intervals. The Rauzy graph is a directed labeled graph, whose vertices are permutations of IET's and the edges lead to permutations obtained by applying one of the operations. Moreover, the edges are labeled by the type of the operation ("a" or "b"). As is well-known, for almost every IET, there is a corresponding infinite path in the Rauzy graph, and the length of the interval on which we induce tends to zero. For any finite "block" of this path, we have a pair of intervals $J \subset I$ and IET's on them, denoted T_I and T_J, such that both are exchanges of m intervals and T_J is the first return map of T_I to J. Let $I_1, \ldots, I_m$ be the subintervals of the exchange T_I and $J_1, \ldots, J_m$ the subintervals of the exchange T_J. Let r_i be the return time for the interval J_i into J under T_I, that is, $r_i = \min\{k > 0 : T_I^k J_i \subset J\}$. Represent I as a Rokhlin tower over the subset J and its induced map T_J, and write

$$I = \bigsqcup_{i=1,\ldots,m,\, k=0,\ldots,r_i-1} T^k J_i.$$

By construction, each of the "floors" of our tower, that is, each of the subintervals $T_I^k J_i$, is a subset of some, of course, unique, subinterval of the initial exchange, and we define an integer $n(i,k)$ by the formula

$$T_I^k J_i \subset I_{n(i,k)}.$$

To the pair I, J we now assign a substitution ζ_{IJ} on the alphabet $\{1, \ldots, m\}$ by the formula

(11.2) $$\zeta_{IJ} : i \to n(i,0)n(i,1)\ldots n(i, r_i - 1).$$

This is the sequence of substitutions arising from the Bratteli-Vershik realization of an IET.

Remark. Veech [35] proved that if the IET satisfies the Keane condition and is uniquely ergodic, then it is uniquely determined by the sequence of renormalization matrices arising from the Rauzy-Veech induction. This implies that the map (11.1) is injective on the set of zippered rectangles of full measure (those which correspond to uniquely ergodic horizontal and vertical flows).

Condition (c) is verified in the next lemma. Words obtained from finite paths in the Rauzy graph will be called admissible.

Lemma 11.1. *There exists an admissible word* $\mathbf{q}$*, which is* **simple***, whose associated matrix* $A(\mathbf{q})$ *has strictly positive entries, and the corresponding substitution* ζ*, with* $Q = \mathsf{S}_\zeta = A(\mathbf{q})^t$ *having the property that there exist* **good return words** $u_1, \ldots, u_m \in GR(\zeta)$*, such that* $\{\vec{\ell}(u_j) : j \leq m\}$ *is a basis of* $\mathbb{R}^m$*.*

We start with a preliminary claim.

Lemma 11.2. *There exists a letter c and an admissible simple word W such that $\eta(j)$ starts with c, for all letters $j \leq m$, where $\eta = \eta(W)$ is the substitution associated to W.*

Proof. Indeed, start with an arbitrary loop V in the Rauzy graph such that the corresponding renormalization matrix has all entries positive. Consider the interval exchange transformation with periodic Rauzy-Veech expansion obtained by going along the loop repeatedly (it is known from [32] that such an IET exists). As the number of passages through the loop grows, the length of the interval forming phase space of the new interval exchange (the result of the induction process) goes to zero. In particular, after sufficiently many moves, this interval will be completely contained in the first subinterval of the initial interval exchange — but this means, in view of (11.2) that $n(i,0) = 1$ for all i, and hence the resulting substitution $\zeta(V^n)$ has the property that $\zeta(V^n)(j)$ starts with $c = 1$ for all j. It remains to make sure that the admissible word is simple. Observe that concatenating two loops V_1, V_2 starting at the same vertex we obtain $\zeta(V_1 V_2) = \zeta(V_2)\zeta(V_1)$. If $\zeta(j)$ starts with c for all j, then $\xi\zeta(j)$ starts with the first letter of $\xi(c)$ for all j. Thus, we can make sure that $V_1 = V^n$ starts and ends with the same Rauzy operation symbol — either a or b, by appending another loop at the end. We can then take V_2 to be the loop of the other Rauzy operation symbol starting at the same vertex. As a result, we obtain the word in the alphabet $\{a, b\}$ corresponding to $W := V_1 V_2$ has the form $a\widetilde{V}_1 ab^k$ or $b\widetilde{V}_1 ba^k$, which is obviously simple. The proof is complete. We are using here the fact that in the Rauzy graph there are both a- and b-cycles starting at every vertex. $\square$

Proof of Lemma 11.1. Let W be the admissible word in the Rauzy graph given by Lemma 11.2. By construction, the matrix S_η of the substitution $\eta = \eta(W)$ has all entries strictly positive, hence $\eta(i)$ contains all letters j, for any $i \leq m$. We can always replace the substitution η by its positive power η^k, since η corresponds to a loop in the Rauzy graph. Note that, for every $i \leq m$, the word $\eta^2(i)$ is a concatenation of *all* words $\eta(j)$, $j \leq m$, in some order, maybe with repetitions, all of which begin with c. Therefore, for every $i \leq m$, the word $\eta^3(i)$ contains every $\zeta(j)$, $j \leq m$, followed by another $\zeta(j')$, also starting with c. It follows that $u_j := \eta(j)$ is a good return word for $\zeta := \eta^3$, for every $j \leq m$. The population vector $\vec{\ell}(\eta(j))$ is the j-th column vector of S_η. As is well-known, the matrices corresponding to Rauzy operations are invertible, which implies that the columns of S_η span $\mathbb{R}^m$. Finally, note that if W is simple, then $\mathbf{q} = WWW$ is simple as well, and it has all the desired properties. The proof is complete. $\square$

The only remaining, key condition to check is (4.5). It will be derived from a variant of the the exponential estimate for return times of the Teichmüller flow into compact sets. For large compact sets of special form, this estimate is due to Athreya [1], whereas in the general form it was established in [8] and independently in [4]. We will mostly use the same notation as above, but indicate the correspondence with the notation of [8]. The symbolic coding of the Rauzy-Veech induction map on the space of interval exchange transformations used in [8] corresponds to the symbolic coding of the Teichmüller flow as a suspension flow over the shift on the space of Markov compacta: indeed, the Rauzy-Veech expansion precisely identifies an interval exchange transformation with a Bratteli-Vershik automorphism (cf. [9]).

The symbol $\Delta(\mathcal{R})$ stands for the space of interval exchange transformations whose permutation lies in a given Rauzy class $\mathcal{R}$ (fixed and omitted from notation); the symbolic space Ω^+ is the one-sided topological Markov chain over a countable alphabet that realizes the symbolic coding of the Rauzy-Veech induction map; the space Ω is its natural extension, the corresponding two-sided topological Markov chain. The space Ω can also be viewed as the phase space of the natural extension of the Rauzy-Veech induction, that is, the space of sequences of interval exchange transformations ordered by nonpositive integers:

$$(\lambda(0), \pi(0)), (\lambda(-1), \pi(-1)), \dots, (\lambda(-k), \pi(-k)), \dots)$$

where $(\lambda(n), \pi(n))$ is the image of $(\lambda(n-1), \pi(n-1))$ under the Rauzy-Veech induction map, in particular

$$\lambda(n) = \frac{A_n \lambda(n+1)}{|(A_n \lambda(n+1))|},$$

where A_n is the renormalization matrix A_n from Section 4. The symbol $|\lambda|$ stands for the sum of coordinates of a vector λ. In other words, the induction map that takes $(\lambda(n), \pi(n))$ to $(\lambda(n+1), \pi(n+1))$ consists in applying the matrix A_n^{-1} and normalizing to unit length. Following [8], we also introduce the "non-normalized" lengths $\Lambda(n)$ inductively by the rule

$$\lambda(0) = \Lambda(0), \quad \Lambda(n) = A_{n+1}\Lambda(n+1) \quad \text{for } n < 0.$$

Informally, $\log |\Lambda(-n)|$ is the "Teichmüller time" corresponding to the discrete normalization "Rauzy-Veech" time n.

Let $\mathbf{q} = q_1 \dots q_k$ be a simple word admissible in the Rauzy class, such that the resulting Rauzy-Veech renormalization matrix has all entries positive. We further let $\Omega_{\mathbf{q}} = [\mathbf{q}.\mathbf{q}]$. The symbol $\mathbb{P}$ denotes the push-forward of the Masur-Veech measure. For $\mathbf{a} \in \Omega_{\mathbf{q}}$, let $\ell_{\mathbf{q}}(\mathbf{a})$ be the return time to the cylinder set $[\mathbf{q}, \mathbf{q}]$ in the negative direction, i.e.

$$\ell_{\mathbf{q}}(\mathbf{a}) = \min\{n \geq 1 : \ \mathbf{a}_{-n-k+1} \dots \mathbf{a}_{-n} = \mathbf{a}_{-n+1} \dots \mathbf{a}_{-n+k} = \mathbf{q}\}.$$

Further, denote by $L_{\mathbf{q}}(\mathbf{a})$ the corresponding "Teichmüller time":

$$L_{\mathbf{q}}(\mathbf{a}) = \log |\Lambda(-\ell_{\mathbf{q}}(\mathbf{a}))|.$$

Proposition 11.3. *There exists $\epsilon > 0$ such that for any $\mathbf{a}^+ \in \Omega_{\mathbf{q}}^+$ we have*

$$\int_\Omega \exp(\epsilon L_{\mathbf{q}}(\mathbf{a})) \, d\mathbb{P}(\mathbf{a}|\mathbf{a}^+) < +\infty.$$

Comparing the definitions, we see that this is equivalent to (4.5), so the remaining condition (d) in Theorem 4.1 will follow from Proposition 11.3.

Proof of Proposition 11.3. We argue in much the same way as in Section 11 of [8] (compare with the arguments involving the "bounded distortion" condition of [5]; see also Section 5.1 of [2]). Our main tool will be Lemma 16 from [8] whose formulation we now recall in our notation:

Lemma 11.4. *For any admissible word $\mathbf{q}$ such that all entries of the matrix $A(\mathbf{q})$ are positive, there exist constants $K_0(\mathbf{q}), p(\mathbf{q})$, depending only on $\mathbf{q}$ and such that the following is true. For any $K \geq K_0$ and any $(\lambda, \pi) \in \Delta(\mathcal{R})$,*

$$\mathbb{P}\left(\exists n : (\lambda(-n), \pi(-n)) \in [\mathbf{q}.\mathbf{q}], \ |\Lambda(-n)| < K \, | \, (\lambda, \pi) = (\lambda(0), \pi(0))\right) \geq p(\mathbf{q}).$$

Remark. In fact, the paper [8] proves this lemma for the coding of the Rauzy-Veech-Zorich induction, that is, for the "Zorich acceleration" of the Rauzy-Veech induction, see [8] for details. However, the proof immediately transfers since it is stated in terms of the "Teichmüller time" $\log |\Lambda(-n)|$, which is invariant under the acceleration. To avoid potential problems with "overlapping occurrences" we choose the word $\mathbf{q}$ to be simple in the Rauzy alphabet $\{a, b\}$, as we do in Lemma 11.2.

The next lemma is a reformulation of Lemma 7 from [8].

Lemma 11.5. *There exists $K_0 \in \mathbb{N}$ and $C = C(\mathcal{R}) \geq 1$ such that the following holds for any $K > K_0$: for all $\mathbf{a}^+ \in \Omega^+$ we have*

$$\mathbb{P}\left(|\Lambda(-1)| > K \, | \, \mathbf{a}^+\right) \leq \frac{C}{K}.$$

Define a random time $k_1(\mathbf{a})$ to be the first moment $n \geq 1$ such that $|\Lambda(-n)(\mathbf{a})| > K$. Note that the map

$$\widetilde{\sigma}(\mathbf{a}) = \sigma^{-k_1(\mathbf{a})}(\mathbf{a})$$

is invertible. Introduce a function $\eta : \Omega \to \mathbb{N}$ by the formula

$$\eta(\mathbf{a}) = \left[\frac{\log |\Lambda(-k_1(\mathbf{a}))|}{\log K}\right].$$

In other words, $\eta(\mathbf{a}) = n$ if $K^n \leq |\Lambda(-k_1(\mathbf{a}))| < K^{n+1}$. Combining Lemmas 11.4 and 11.5 we obtain

Proposition 11.6. *For any $\mathbf{a}^+ \in \Omega^+$ and any $r \in \mathbb{N}$ we have*

$$\mathbb{P}\left(\{\mathbf{a} : \ \eta(\mathbf{a}) = r, \ \mathbf{a}_{-k_1(\mathbf{a})+1}, \ldots, \mathbf{a}_0 \ \text{does not contain } \mathbf{q} \, | \, \mathbf{a}^+\}\right) \leq \frac{C}{K^{r-1}} \cdot (1 - p(\mathbf{q})).$$

Now, take a large N and let

$$n_N(\mathbf{a}) = \min\{n : \ \eta(\mathbf{a}) + \cdots + \eta(\widetilde{\sigma}^n \mathbf{a}) \geq N\}.$$

Let also

$$k_n(\mathbf{a}) = k_1(\mathbf{a}) + \cdots + k_1(\widetilde{\sigma}^n \mathbf{a}).$$

Then we have, by the definition of η:

$$(11.3) \qquad\qquad K^N \leq |\Lambda(-k_{n_N(\mathbf{a})})| < K^{n_N(\mathbf{a}) + \eta(\mathbf{a}) + \cdots + \eta(\widetilde{\sigma}^n \mathbf{a})}.$$

Clearly $n_N(\mathbf{a}) \leq N$, and observe that for all $\mathbf{a}^+ \in \Omega^+$,

$$\mathbb{P}\left(\mathcal{B}(N) \,|\, \mathbf{a}^+\right) \leq \frac{C}{K^N}, \quad \text{where } \ \mathcal{B}(N) = \{\mathbf{a} : \ \eta(\widetilde{\sigma}^{n_N(\mathbf{a})}\mathbf{a}) > N\},$$

by Lemma 11.5. Consider the set

$$\widetilde{\Omega}(N) = \{\mathbf{a} : \ \mathbf{a}_{-k_{n_N(\mathbf{a})}+1}, \ldots, \mathbf{a}_0 \ \text{does not contain the word } \mathbf{q}\} \cap \{\mathbf{a} : \ \eta(\widetilde{\sigma}^{n_N(\mathbf{a})}\mathbf{a}) \leq N\}.$$

Note that

$$\{\mathbf{a} : L_{\mathbf{q}}(\mathbf{a}) > 3N\} \subset \widetilde{\Omega}(N) \cup \mathcal{B}(N).$$

It suffices, therefore, to prove that there exists $\rho < 1$ such that

$$\mathbb{P}\left(\widetilde{\Omega}(N) \,|\, \mathbf{a}^+\right) \leq \rho^N.$$

We have from Proposition 11.6, for any $\ell \in \mathbb{N}$ and $n_1, \ldots, n_\ell \in \mathbb{N}$:

$$\mathbb{P}\left(\eta(\mathbf{a}) = n_1, \ \eta(\widetilde{\sigma}\mathbf{a}) = n_2, \ldots, \eta(\widetilde{\sigma}^\ell \mathbf{a}) = n_\ell, \ \mathbf{a}_{-k_\ell(\mathbf{a})+1}, \ldots, \mathbf{a}_0 \ \text{does not contain } \mathbf{q} \,|\, \mathbf{a}^+\right)$$

$$(11.4) \quad \leq (1 - p(\mathbf{q}))^\ell \left(\frac{C}{K}\right)^{(n_1 + \cdots + n_\ell) - \ell}.$$

Choose $K \in \mathbb{N}$ such that $1 - p(\mathbf{q}) + \frac{C}{K} < 1$. We have

$$\mathbb{P}\left(\widetilde{\Omega}(N) \,|\, \mathbf{a}^+\right) \ \leq \ \sum_{\widetilde{N}=N}^{2N} \mathbb{P}\left(\widetilde{\Omega}(N) \,|\, \mathbf{a}^+ \ \& \ \eta(\mathbf{a}) + \cdots + \eta(\widetilde{\sigma}^{n_N(\omega)}\mathbf{a}) = \widetilde{N}\right)$$

$$\leq \ \sum_{\widetilde{N}=N}^{2N} \left(1 - p(\mathbf{q}) + \frac{C}{K}\right)^{\widetilde{N}} \leq \rho^N,$$

using (11.4) and the binomial formula in the last line, and Proposition 11.3 is proved. This concludes the proof of Theorem 1.1 as well. $\qquad\square$

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